课程目标.

基本概念：Algebraic varieties, dimension, smooth point/singular point. Coordinate ring, tangent space. map/isomorphism, intersection number, blow up, divisor

① 判断光滑点/奇点

② 不可约分解

③ 判断代数簇同构

考试：具体问题

读书报告 可能加分

考核 平时作业 40%

期末考试 60%

课堂思考题 +

自己思考 +

教材: Fulton

Mumford

本课程是要研究交代数几何

~~代数簇~~

代数簇之间的映射

$f: X \to Y$.

$\dim X = d \quad \dim Y = d'$

$d < d' \Rightarrow f$ not surj.

$d \geq d'$, f surj.

$\Rightarrow \dim f^{-1}(y_0) = d' - d$.

对"大部分 y_0"

$d = d$, f surj

$\Rightarrow f^{-1}(y_0) = k$ 对"大部分 y_0"

不固定

相交. $X^d, Y^{n-d} \subseteq \mathbb{P}^n$.

$X \cap Y$ finite -

引入代数几何.

线性代数.

$$AX = b$$

$$\dim \ker V = n - \operatorname{rank} A.$$

线性方程 → 多项式方程.

$$V(I) = \{ x \mid f(x) = 0, \forall f \in I \}.$$

Def 1. Zariski topo.

the closed algebraic set in $\mathbb{C}^1$

X is vanishing points of finite many polys.

Noetherian

第一章 代数簇 (Algebraic variety)

几何与代数对应

不可约分解

光滑点/非光滑点 维数.

坐标环. (Cooperate ring)

局部环. (local ring)

1. 代数集(多项式公共零点集).

Definition 1.

Closed algebraic set of $\mathbb{C}$ is

$$X = \{x \mid f_i(x) = 0, \forall i = 1, \cdots, n\} = \bigcap_i V(f_i)$$

Denotes by $V(f) := \{p \in \mathbb{C}^n, f(p) = 0\}$.

If f is non-constant, call $V(f)$

hypersurface.

$$V(f_1, \cdots, f_n) := \{p \mid f_i(p) = 0, p = 1, \cdots, n\}.$$

So every algebraic set is finite intersection of some hypersurfaces.

$k[x_1, \cdots, x_n]$ is Noetherian ring.

A set $X \subseteq \mathbb{C}^n$ is called closed affine algebraic set, if $\exists S$. s.t.

$X = V(S)$

By Noetherian property, this is equivalent to $X = V(f_1, \cdots, f_n)$.

Tangent space

$\mathbb{F}_p$

Zariski tangent space

$$\left\{ X = \{x_1, \cdots, x_n\} \in \mathbb{C} \;\middle|\; \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(a)(X_i - a_i) = 0, \ \forall f \in P \right\}$$

denote as $T_{x,a}$.

$T_{x,a}$ naturally become the vector space with origin at a.

Suppose $P = (f_1, \cdots, f_r)$

$$\Rightarrow T_{x,a} = \{ x : \sum_{i=1}^{n} \frac{\partial f_\alpha}{\partial X_i}(a)(X_i - a_i) = 0, \ \alpha = 1, \cdots, r \}$$

$$\dim_{\mathbb{C}} T_{x,a} = n - \operatorname{rank}\left(\frac{\partial f_\alpha}{\partial f_i}\right)_{\substack{1 \le \alpha \le r \\ 1 \le i \le n}}$$

$$\Rightarrow \{ a \in X : \dim T_{x,a} \geq k \}$$

is algebraic set.

{compute all $(n-k+1)\times(n-k+1)$-minors

of $\left(\frac{\partial f_\alpha}{\partial f_i}\right)_{\alpha, i}$}

上半连续: $u\subseteq\mathbb{R}\to\mathbb{R}$

$\limsup_{y\to x} f(y)\leq f(x)$

$\Leftrightarrow \{x\in u \mid f(x)\geq\beta\}$ is closed

经典拓扑

Zariski topology: $\forall\beta$, if

$f^{-1}([\beta, +\infty))$ is closed.

(under Zariski topology)

Consider function f

$$f: X \longrightarrow \mathbb{Z}_{\geq 0} \subseteq \mathbb{R}$$

$$f(a) = \dim T_{X,a}$$

By the discussion above

f is upper-continuous.

切空间的内蕴刻画.

Coordinate ring

$$\gamma(X) = \mathbb{C}[X_1, \dots, X_n]/P$$

derivation of X at a is a function D.

$D : \gamma(X) \longrightarrow \mathbb{C}$ satisfying:

- D is $\mathbb{C}$-linear
- Leibniz's rule:

$$D(fg) = D(f)\,g(a) + D(g)\,f(a)$$

- $D(\alpha) = 0, \quad \forall \alpha \in \mathbb{C}$

It's easy to check

$$\mathcal{O}(X) = \mathbb{C}[X_1, \dots, X_n] / P$$

derivation at a $D: \mathcal{O}(X) \to \mathbb{C}$

$\Leftrightarrow$ $\mathbb{C}[X_1, \dots, X_n]$'s derivation

$D': \mathbb{C}[X_1, \dots, X_n] \to \mathbb{C}$. s.t.

$D'(f) = 0, \ \forall f \in P$

$\Leftrightarrow$ $D'\big|_P = 0$

$$
\begin{tikzcd}
\{\text{derivation of } \mathbb{C}[X_1,\dots,X_n] \text{ at } a\} \arrow[d, shift right] & D' \arrow[d, mapsto] & D \\
\mathbb{C}^n \arrow[u, shift right] & (\lambda_1,\dots,\lambda_n) & (\lambda_1,\dots,\lambda_n) \arrow[u, mapsto]
\end{tikzcd}
$$

Hence

$$
\begin{tikzcd}
\{\gamma(X) \text{ derivations}\} \arrow[d, "\wr"] \\
\left\{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n \;\middle|\; \sum_{i=1}^{n}\frac{\partial f}{\partial X_i}(a)\lambda_i=0,\ \forall f\right\} \arrow[d, "\wr"] \\
T_{X,a}
\end{tikzcd}
$$

Now introduce the third definition of tangent space.

$X = V(p)$ be algebraic variety

Local ring of X at a is

$$\mathcal{O}_{a,X} = \left\{ \frac{f(x_1,\cdots,x_n)}{g(x_1,\cdots,x_n)} \;\middle|\; g(a) \neq 0 \right\}$$

$\mathcal{O}_{a,X}$ = Localization of $\mathcal{J}(X)$

at $S = \{ f \mid f(a) \neq 0 \}$.

$= \{$ function germ $f: X \to \mathbb{C} \mid \exists a \in U \subset X,$

u open, s.t. $\phi|_u = \frac{f}{g}\Big|_u$, $\frac{f}{g} \in \mathbb{C}(X, \cdot)$

Zariski topology $\{ g(a) \neq 0 \}$

function germ of X at a =

a function $\phi: u \to \mathbb{C}$, u open

$(\phi, u) = (\phi', u')$, if $\exists V \subseteq u \cap u'$ open,

$$\phi|_V = \phi'|_V$$

Semi-local ring: A ring with finite many maximal ideal

Definition: regular function.

Suppose $\phi : X \to \mathbb{C}$ is regular at a,

(germ). , if:

- $\exists\, U \subseteq X$ open,
- $\exists\, F, G \in \mathbb{C}[X_1, \dots, X_n]$ s.t.

$G(y) \neq 0$, $\forall y \in U$, and

$$\phi\Big|_U = \frac{F}{G}\Big|_U$$

The ring of regular function is tangent space.

It's easy to check these definitions are equivalent

$\mathcal{O}_{a,X} = \{$ regular functions (germs) at $a \}$

$\mathcal{O}_{a,X}$ is a local ring, its maximal ideal $m_a = \{ \varphi \mid \varphi(a) = 0 \}$

residue field $= \mathbb{C}$

fn. field

$$\mathcal{O}(X) \subseteq \mathcal{O}_{a,X} \subseteq K \leftarrow \text{fraction field}$$

every derivation $D: \mathcal{O}(X) \to \mathbb{C}$

extends uniquely to $\mathcal{O}_{a,X}$:

$$D(f/g) = \frac{Df \cdot g(a) - Dg \cdot f(a)}{g^2(a)}$$

$\Rightarrow T_{X,a} = \{$ all derivations of $\mathcal{O}_{a,X}$

centered at $a \}$.

Definition: differential

$f \in \mathcal{O}_{a,X}$. differential of f at a

is a linear map $df : T_{X,a} \to \mathbb{C}$

$df(D) = D(f)$

View $\mathcal{O}_{a,X} \subseteq K$

Theorem. $\bigcap_{a \in X} \mathcal{O}_{a,X} = \gamma(X)$

Proof:

See commutative algebra.

Recall:

Tangent space

$= \{ \mathbb{C}\text{-derivation } D : \mathcal{O}_{a,X} \to \mathbb{C} \}$, Residue field. $\mathcal{O}_{a,X}/m \xrightarrow{\sim} \mathbb{C}$

$D: \mathcal{O}_{a,X} \to \mathbb{C}$ satisfying:

(1) $D(\text{constant}) = 0$.

$\Rightarrow D(f) = D(f(a) + (f - f(a))) = D(f - f(a))$.

D is uniquely determined by its value on m_a.

(2) $D(m_a^2) = 0$

D is uniquely determined by its value on m_a/m_a^2

(3): the induced map of D on m_a/m_a^2 is $\mathbb{C}$ - linear.

(4) Conversely, every $\mathbb{C}$-linear map from $m_a/m_a^2 \to \mathbb{C}$ determined a derivation.

$\Rightarrow T_{X,a}$ = dual space of m_a/m_a^2

m_a is the image of $[X_1-a_1, \cdots, X_n-a_n]$ in $\mathcal{O}(X)$ or $\mathcal{O}_{X,a}$

Definition. $X = V(P) \subseteq \mathbb{C}^n$ algebraic variety, $a \in X$, m_a is the maximal ideal of $\mathcal{O}(X)$ corresponds to a.

Call m_a / m_a^2 Zariski cotangent space at a

Dimension, smooth pts, singular pts.

Definition $X = V(P)$

$\mathcal{O}(X)$, $\mathbb{C}(X) = \mathrm{Frac}(\mathcal{O}(X))$

The dimension of X is

$$\dim_{\mathbb{C}} X := \mathrm{tr.d}_{\mathbb{C}}\, \mathbb{C}(X)$$

tr.d.: transcendental degree.

Cardinality of the maximal algebraic independent subset.

$a \in X$, a is smooth point of

$\dim X = \dim_{\mathbb{C}} T_{X,a}$

Or, call a singular pt.

If every point of X is smooth, call

X smooth variety.

Or call X singular variety.

(non-smooth variety)

Theorem. $X = V(P)$.

The following statements is true:

(1) $\forall a \in X$, $\dim T_{X,a} \geq \dim X$

(2) $\exists$ non-empty open subset U of X,

every pts in U is smooth.

(roughly, most of X is smooth).

(3) $\dim_{\mathbb{C}} X = \min\limits_{a \in X} (\dim T_{X,a})$

proof: step 1. algebraic pre-knowledge.

K/k separable extension, then

$\mathrm{tr.d}_k K = \dim_K$ [all k-derivation : $K \to K$)

Hence

$\dim X = \mathrm{tr.d}_{\mathbb{C}} \mathbb{C}(X) = \dim_{\mathbb{C}(X)}$ ($\mathbb{C}$-derivation D: $\mathbb{C}(X) \to \mathbb{C}(X)$)

$= \dim_{\mathbb{C}(X)}$ ($\mathbb{C}$-derivation D: $\mathcal{O}(X) \to \mathbb{C}(X)$)

$= \dim_{\mathbb{C}(X)}$ ($\mathbb{C}$-derivation D: $\mathbb{C}[x_1, \dots, x_n]$ $\to \mathbb{C}(X)$ which kills P).

$= \dim_{\mathbb{C}(X)}$ (n-tuples $(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}(X))^n$ such that $\sum \frac{\partial f}{\partial x_i} \lambda_i = 0$, $\forall f \in P$)

$P = (f_1, \dots, f_r)$

$= \dim_{\mathbb{C}(X)} \left((\lambda_1, \dots, \lambda_n) \in \mathbb{C}(X)^n, \sum \frac{\partial f_j}{\partial x_i} \lambda_i = 0, \forall j \right)$

Let $A = \left(\frac{\partial f_j}{\partial X_i}\right)_{j,i} \in C(X)^{n \times l}$

$\Rightarrow \dim X = n - \operatorname{rank}_{\mathbb{C}(X)} A$

$\dim T_{X,a} = n - \operatorname{rank}_{\mathbb{C}} A(a)$

Check each minors

$\Rightarrow \operatorname{rank} A \geq \operatorname{rank} A(a).$ $\Rightarrow$ (1) ✓.

(2) : Suppose $\operatorname{rank}_{C(X)} A = r$

$\Rightarrow$ Suppose all n-minors are $g_1 \sim g_s$

$\Rightarrow \operatorname{Sing}(X) = X \cap V(g_1, \cdots, g_s)$

$\uparrow$

(3) : Immediately from (1), (2)

Denotes $\mathrm{Sing}(X) = \{a \in X, \dim T_{X,a} \geq \dim X + 1\}$

Definition. If $X = V(I) \subseteq \mathbb{C}^n$

$X = \bigcup_{i=1}^{r} X_i$ is its irre. decomposition.

$a \in X$, a smooth in $X \Leftrightarrow \begin{cases} (1) \exists! \, i, \text{ s.t. } a \in X_i \\ (2) \, a \in X_i \text{ smooth.} \end{cases}$

e.g. $\dim \mathbb{C}^n := \mathrm{tr.d}\, \mathbb{C}(x_1, \cdots, x_n) = n$

$\dim(\{\text{single pt}\}) = \dim\{a\}$

$= \mathrm{tr.d}\, \mathbb{C}$

$= 0.$

e.g. $X = \{(a^3, a^2), a \in \mathbb{C}\} = V(X_1^2 - X_2^3)$

Claim: $\dim X = 1$

(1) $T_{X,(0,0)} = \{X = (X_1, X_2) \mid 0X_1 = 0, 0X_2 = 0\} = \mathbb{C}^2$

(2) $T_{X,(a^3,a^2)} = \left\{X = (X_1, X_2) \,\middle|\, \begin{array}{l} 2a(X_1 - a^2) \\ -3a^2(X_1 - a^3) = 0 \end{array} \right\}$

$a \neq 0$

$\cong \mathbb{C}$

Hypersurface.

$X = V(f) \subseteq \mathbb{C}^n$. f irreducible.

$\dim X = \mathrm{tr.d.}_{\mathbb{C}}(\mathrm{Frac}(\mathbb{C}[x_1, \cdots, x_n]/f))$

$= n-1$

$$\mathrm{Sing}(X) = \{a \in X \mid T_{X,a} = \mathbb{C}^n\}$$

$$= \{a \in X \mid \frac{\partial f}{\partial x_i}(a) = 0,\ \forall i\}.$$

Theorem.

$X, Y \subseteq \mathbb{C}^n$ algebraic varieties

$X \subsetneq Y$

$\Rightarrow \dim X < \dim Y$

Theorem.

X, Y algebraic varieties

$X \subsetneq Y$

$\Rightarrow \dim X < \dim Y$

Pf: $X \quad P_X \quad \operatorname{trd} \gamma(X)$

$Y \quad P_Y \quad \operatorname{trd} \gamma(Y)$

$P_Y \subsetneq P_X$

$\Leftrightarrow \quad \underset{\substack{\neq \\ 0}}{P_X/P_Y}$ in γ_Y prime.

Lemma. $\mathbb{C} \hookrightarrow A$, A integral

$P \subseteq A$ prime

$\Rightarrow \operatorname{tr.d}_{\mathbb{C}} A \geq \operatorname{tr.d}_{\mathbb{C}} A/P$

The equality holds if and only if $P=\{0\}$.

Clearly it's enough if we pf the Lemma.

pf of lemma:

Suppose $\operatorname{tr.d}_{\mathbb{C}} = n$, $P \neq \{0\}$.

If this statement is false, then $\operatorname{tr.d}_{\mathbb{C}} A/P \geq n$, $\exists$ n elements algebraic independent.

$A \to A/P$ surjective

$x_1 \sim x_n$ $\bar{x}_1 \sim \bar{x}_n$ algebraic independent.

$\exists f \in P, f \neq 0$

$\Rightarrow \exists G \in \mathbb{C}[T_0, \dots, T_n]$

s.t. $G(f, x_1, \dots, x_n) = 0$

$\Rightarrow G(\bar{f}, \dots, \bar{x}_n) = 0$ $\bar{f} = 0$

$\Rightarrow G(0, T_1, \dots, T_n) = 0.$

$\Rightarrow \quad T_0 \mid G$

A is integral domain, $\mathbb{C}[T_0, \dots, T_n]$ is UFD.

$\Rightarrow$ We can assume G is irreducible

$\Rightarrow \quad G = \alpha T_0, \ \alpha \in \mathbb{C}, \ \alpha \neq 0$

$\Rightarrow \quad f = 0$, Contradiction!

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(1) for $X = V(F)$

$$X = \bigcup_{i=1}^{r} X_i$$

algebraic set.

X

(2) $Y = V(P)$

$Y = \mathrm{Sm}(Y) \sqcup \mathrm{Sing}(Y)$

$\mathrm{Sing}(Y) = \bigcup_{i=1}^{s} Y_i \,,\ Y_i \subsetneq Y$

$\Rightarrow \dim Y_s < \dim Y$

Hence, from an algebraic set, we finally get:

$X = V(I) = \bigcup_{i=1}^{r} X_i \rightsquigarrow \bigcup_{i=1}^{r} \mathrm{Sing}(X_i)$

strictly small than X.

Via (1), (2).

s.t. every irreducible components is smooth

$\Rightarrow X = \bigcup_{i=1}^{N} U_i$

U_i is set of smooth pts of some algebraic varieties

Specially, when $X = V(P)$ is a variety.

- $U = \mathrm{Sm}(X)$
- $X \setminus U = X_1^{(1)} \cup \cdots \cup X_k^{(1)}$

$$\dim X_i^{(1)} \le \dim X - 1$$

$$u_i^{1} = \mathrm{Sm}\,(X_1^{(1)})$$

- $$X = (u \cup \cdots \cup u_k^{(1)})$$

$$= \bigcup_{i=1}^{k} (X_i^{(1)} \setminus u_i^{(1)})$$

$$= \bigcup_{i=1}^{k} \mathrm{Sing}(X_i^{(1)})$$

$$= u_1^{(2)} \cup \cdots \cup u_l^{(2)}$$

Let $u_i^{(2)} = \mathrm{Sm}(X_i^{(2)})$

- X is expressed as the union of "smooth pts"

Definition. $X = V(I) \subseteq \mathbb{C}^n$ is algebraic set. Call a subset $U \subset X$ is locally closed, if U in $\overline{U} \subseteq X$ is open.

Equivalently, U can be expressed as (open $\cap$ closed) in X.

The decomposition above is

$$X = \bigcup_{i,j} U_i^{(j)} \qquad U_i^{(j)} = Sm(X_i^{(j)})$$

$X_i^{(j)}$ is irreducible

$\Rightarrow U_i^{(j)}$ is dense in $X_i^{(j)}$

(because $X_i^{(j)} = \overline{U_i^{(j)}} \cup \mathrm{Sing}(X_i^{(j)})$)

$\Rightarrow U_i^{(j)}$ is locally closed in $X_i^{(j)}$

Combine these, an algebraic variety has a stratification, become the finite union of smooth locally closed set

$$X = \bigcup_{i,j} U_i^{(j)}$$

1. Corollary

n-1 dimensional subvariety of $\mathbb{C}^n$ is hypersurface

pf: Suppose $X = V(P)$, $\dim X = n-1$

$\exists g \in P$, $g \neq 0$, $g \in \mathbb{C}[X_1, \cdots, X_n]$

$$g = \prod_{i=1}^{s} g_i^{k_i}$$

$$g\big|_X = 0 \iff X \subseteq V(g) = V(g_1) \cup \cdots \cup V(g_s)$$

X irreducible $\Rightarrow \exists i,\ X \subseteq V(g_i)$

Claim: $X = V(g_i)$

this is because $\dim X = n-1 \geq \dim V(g_i)$

2. Smooth point.

Proposition 1. $\mathcal{O}_{a,X}$ is UFD

Proposition 2. $\mathrm{Sm}(X)$ is complex manifold.

pf of proposition 1:

a smooth

$$\Rightarrow \underset{\substack{\parallel \\ \mathrm{tr.d}_{\mathbb{C}} \, \mathbb{C}(X)}}{\dim X} = \dim T_{x,X} = \dim \left(\frac{m_a}{m_a^2}\right)^*$$

$$\mathrm{tr.d}_{\mathbb{C}} \, \mathbb{C}(X) = \mathrm{tr.d}_{\mathbb{C}} \, \mathcal{O}_{a,X}$$

Definition. regular local ring ($\mathbb{C}$).

If $\boxed{\dim A} = \left(\frac{m}{m_a^2}\right)$, call A a regular local ring

$$a \in Sm(X) \Leftrightarrow \mathcal{O}_{a,X} \text{ is regular}$$

Theorem.

Every regular local ring is UFD.

Counter example:

$$X = V(x_1^2 + \cdots + x_n^2) \subseteq \mathbb{C}^n, \quad n \geq 5$$

$\Rightarrow$ $\mathcal{O}_{0,X}$ is UFD, but 0 is a singular pt.

A complex manifold of dimension n is:

(1) a topology space

(2) $M = \bigcup_\alpha U_\alpha$

$\exists$ $\varphi_\alpha : U_\alpha \to V_\alpha$, $V_\alpha \subseteq \mathbb{C}^n$ is open,

φ_α is homeomorphism.

s.t. $\forall \alpha, \beta$, $V_\alpha \cap V_\beta \neq \emptyset$, and

$\varphi_\alpha \quad V \subseteq \mathbb{C}^n$

$$\begin{tikzcd}
 & V_\alpha \subseteq \mathbb{C} \arrow[d, dashed, "\varphi_\beta \circ \varphi_\alpha^{-1} \text{ is holomorphic}"] \\
U_\alpha \cap U_\beta \arrow[ur, "\varphi_\alpha"] \arrow[r, "\varphi_\beta"'] & V_\beta
\end{tikzcd}$$

$$X = V(I) = \bigcup_{i=1}^{r} X_i$$

$$\Rightarrow \bigcup_{i=1}^{r} \mathrm{Sing}(X_i)$$

Dimension will decrease at least 1 at each step.

$$X = \bigcup_{i=1}^{N} U_i$$

U_i is locally closed.

$X = V(P) \subseteq \mathbb{C}^n$. is an algebraic variety

(1) $a \in X$ smooth $\Rightarrow \mathcal{O}_{a,X}$ regular local ring $\Rightarrow$ UFD

(2) $S_m(X)$ is submanifold

$(U_\alpha, \varphi_\alpha)$. $\varphi_\alpha : U_\alpha \to V_\alpha$ homeomorphism.

$\Downarrow$

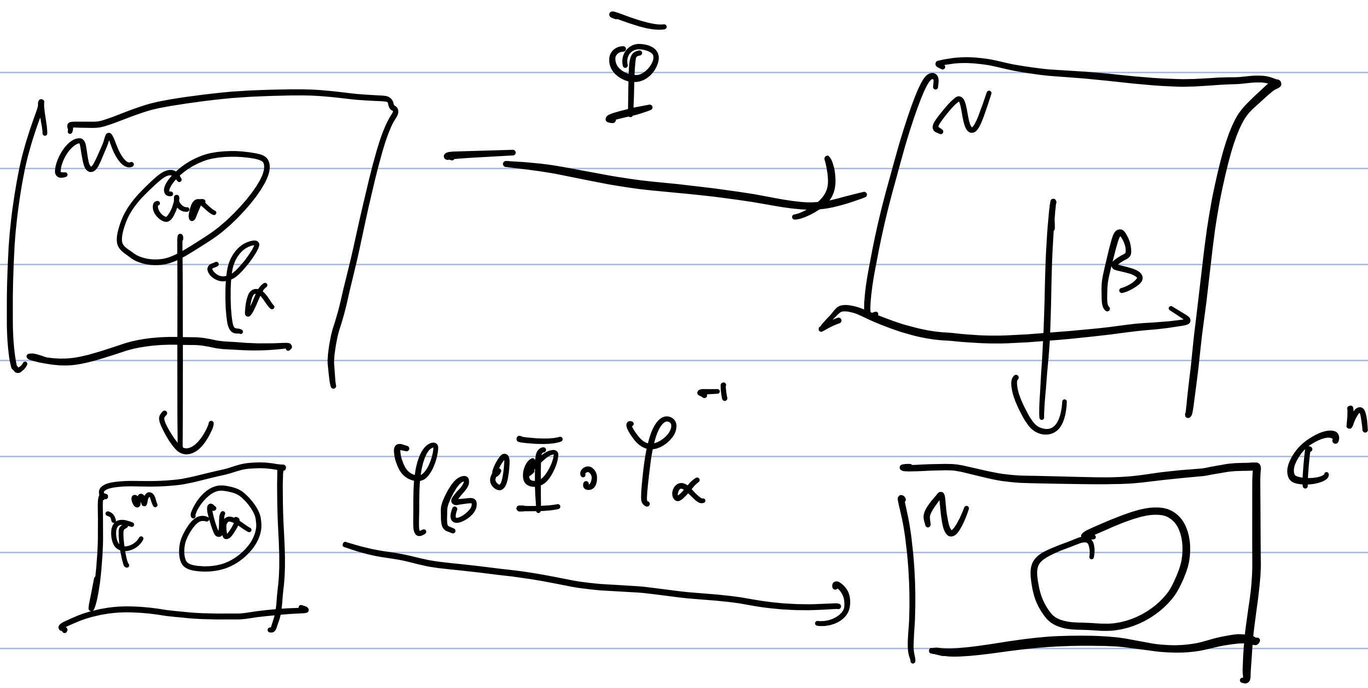
chart (坐标系)

$\{(U_\alpha, \varphi_\alpha)\}$ coordinate atlas (坐标集)

The maps between complex manifold:

M^m, N^n complex

$\Phi : M \to N$ is a map, s.t.



Holomorphic isomorphism:

If Φ is bijective.

Example.

(1) $\mathbb{C}^n$, $Id: \mathbb{C}^n \to \mathbb{C}^n$.

(2) open subset $V \subseteq \mathbb{C}^n$, $Id: V \to V$

(3) $\mathbb{CP}^n = \mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^*$

$U_i := \{ [z_0, \dots, z_n] \in \mathbb{P}^n, z_i \neq 0 \}$.

$\varphi_i : U_i \to \mathbb{C}^n$

$[x_0, \dots, x_n] \to [\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots]$

(4). M complex manifold, $V \subseteq M$ open

$\Rightarrow$ V complex manifold.

Coordinate system.

M complex manifold, $p \in M$. coordinate chart/system at p is

(1): $U \ni p$, $U \subseteq M$, $V \subseteq \mathbb{C}$ open

(2) $\varphi: U \to V$ holomorphic isomorphism.

Subcomplex.

M is a complex manifold

$S \subseteq M$ subset, we call S as a

k- dimension subcomplex, if $\forall p \in S$

$\exists$ Coordinate chart $(U, \varphi) \subset M$, $p \in U$

$\varphi(p) = 0$, s.t.

$$\varphi(S \cap U) = \{(z_1, \cdots, z_n) \in \varphi(U) : \underset{\overset{\shortparallel}{0}}{z_1 = \cdots = z_{n-k}}\}$$

3. $Sm(X) \subseteq \mathbb{C}^n$ subcomplex, $X = V(p)$

let $d = \dim X$

$\Rightarrow \forall a \in Sm(X)$, $\exists$ $a \in U$ [classical topology]

and $\exists \varphi: U \to \varphi(U) \subset \mathbb{C}^n$, $\varphi(a) = 0$

s.t. $\varphi(S \cap a) = \{(z_1, \cdots, z_{n-d}, \cdots) \mid z_i = 0, \forall i \leq n-d\}$

First, Smooth (X) is open

Suppose $\mathcal{O} = p \in \mathbb{C}$

(4.1) Algebraic knowledge

Complement: Suppose A is a Noetherian ring, $I \subseteq A$ ideal

Natural projection: $A/I^{n+1} \to A/I^{n}$.

$$\bar{A} = \varprojlim_{k} A/I^{k}$$

I-adic.

Flatness.

A, B are two rings, $\phi: A \to B$ is homemorphism. B is flat if

$\otimes B$ is exact

faithfully flat: if

$\otimes B$ is exact and faithful exact

functor. i.e.:

$M' \to M \to M''$ exact

$\Leftrightarrow M' \otimes_A B \to M \otimes_A B \to M'' \otimes_A B$ exact

Proposition. assume $a = 0$

- $\mathcal{O}_{0,\mathbb{C}^n}$ is $\mathbb{C}[X_1,\cdots,X_n]$-flat

(localization is exact).

- (A, m_A) Noetherian local ring,

  $\hat{A}$ is its complement

$\Rightarrow$ $\hat{A}$ is A-faithfully flat

$\Rightarrow$ $\mathbb{C}[[X_1,\cdots,X_n]] = \widehat{\mathcal{O}_{0,\mathbb{C}^n}}$ is

$\mathcal{O}_{0,\mathbb{C}}$ - faithfully flat.

- $\phi: A \to B$ is faithfully flat

  (take B as A-module)

$\Rightarrow \forall I \subseteq A$

$IB \cap A := f^{-1}(IB) = I$

$\mathbb{C}[X_1, \cdots, X_n] \hookrightarrow \mathcal{O}_{0, \mathbb{C}^n} \hookrightarrow \mathbb{C}[[X_1, \cdots, X_n]]$

$P = (f_1, \cdots, f_r)$, $f_i \in \mathbb{C}[X_1, \cdots, X_n]$

Suppose $a \in Sm(X)$, $\operatorname{rank} \frac{\partial(f_1, \cdots, f_r)}{\partial(X_1, \cdots, X_n)} = d$

$\Rightarrow$ (1) Suppose $a = 0$.

(2) Suppose $\frac{\partial(f_1, \cdots, f_r)}{\partial(X_1, \cdots, X_n)} = \begin{bmatrix} I_{n-d} & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} f_1 = X_1 + O(|X|^2) \\ f_2 = X_2 + O(|X|^2) \\ \vdots \\ f_{n-d} = X_{n-d} + O(|X|^2) \end{cases}$$

f_{r-d}

$\vdots$

$f_n = O(|X|^2)$

Case 1: $r = n-d$

$(X_1, \dots, X_n) \xrightarrow{\Phi} (f_1(X_1, \dots, X_n), \dots, f_r, X_{r+1}, \dots, X_n)$

$(D\Phi)(0) = Id$

Case 2.

$r > n-d$

$\mathfrak{a}$ $\qquad \mathfrak{a}' \qquad \mathfrak{a}''$

$\mathbb{C}[X_1, \dots, X_n] \hookrightarrow \mathcal{O}_{0,\mathbb{C}^n} \hookrightarrow \mathbb{C}[[\dots]]$

$\mathfrak{a}'' \subseteq \mathbb{C}[[X_1, \dots, X_n]]$ prime

$\Rightarrow \mathbb{C}[[X_1, \dots, X_n]] / \mathfrak{a}'' (\sim)$

$$\mathbb{C}[[X_{n-d+1}, \cdots, X_n]]$$

$$\mathbb{C}[X_1, \cdots, X_n]/\mathfrak{a} \hookrightarrow \mathcal{O}_{0,\mathbb{C}^n}/\mathfrak{a}' \hookrightarrow \mathbb{C}[[X_1, \cdots, X_n]]/\mathfrak{a}''$$

$$= \mathbb{C}[[X_{n-d+1}, \cdots, X_n]]$$

See mumford. 1.19.

$\mathbb{C}[X_1, \cdots, X_n]$

$\mathfrak{p} = (f_1, \cdots, f_r)$ prime

$\mathfrak{a} = \mathfrak{a}' \cap \mathbb{C}[X_1, \cdots, X_n]$

$$\mathfrak{p} = (f_1, \cdots, f_r)$$

$$\cup\!\!|$$

$$I = (f_1, \cdots, f_{n-d})$$

$$\supset\!\!|\!|$$

$$\mathfrak{a} = \mathfrak{a}' \cap \mathbb{C}[X_1, \cdots, X_n]$$

$$X = V(\mathfrak{p})$$

$$Y = V(\mathfrak{a})$$

$$\gamma(Y) = \mathbb{C}[X_1, \cdots, X_n]/\mathfrak{a} \hookrightarrow \mathbb{C}[\overline{X}_{n-d+1}, \cdots, X_n]$$

$\overline{X_{n-d+1}}, \cdots, \overline{X_n}$ is algebraic independent.

$\Rightarrow \dim Y \geq d$

$\mathfrak{a} \subseteq \mathbb{C}[x_1, \cdots, x_n]$

Suppose $\mathfrak{a} = (\tilde{f}_1, \cdots, \tilde{f}_s)$, $\tilde{f}_i \in \mathbb{C}[x_1, \cdots, x_n]$

Notice that

$\mathfrak{a} = \mathfrak{a}' \cap \mathbb{C}[x_1, \cdots, x_n] = (f_1, \cdots, f_{n-d}) \mathcal{O}_{0,\mathbb{C}^n} \cap \mathbb{C}[x_1, \cdots, x_n]$

$= \{ f = \sum_i \frac{h_i f_i}{k_i}, \ k_i \neq 0 \}$

Let $K = \prod_i k_i$

$\Rightarrow \exists K_i$, s.t. $K_i \tilde{f}_1 \in I = (f_1, \cdots, f_{n-d})$

$\Rightarrow \quad K \tilde{f}_1 \in I \Rightarrow K \mathfrak{a} \subseteq I$.

Suppose $V(I) = \bigcup_{i=1}^{\sim} V_i$

irreducible decomposition.

$$T_{V_i,0} \subseteq \{ X = (X_1, \cdots, X_n) \in \mathbb{C}^n, \sum_j \frac{\partial f_\alpha}{\partial X_j} = 0, \ \forall \alpha = 1, \cdots, n-d \}$$

$$\frac{\partial(f_1, \cdots f_{n-d})}{\partial(X_1, \cdots, X_n)} = (I_{n-d}, 0)$$

$\Rightarrow \dim T_{V_i,0} \leq d$

$\Rightarrow \dim V_i \leq d$

$$Y \subseteq V(I) = \bigcup_{i=1}^{N} V_i$$

Y irreducible $\Rightarrow Y \subseteq V_i$

$\Rightarrow \dim Y \leq d$

$\Rightarrow \dim Y = d$

Next. find P.

$X \subseteq V(I) = Y \cup V'$

$0 \in X$, $0 \notin V'$

$\Rightarrow X \subseteq Y$

$\dim X = d = \dim Y$

$\Rightarrow X = Y$

$$\Rightarrow \boxed{V(I) = V(f_1, \dots, f_{n-d}) = X \cup V' \quad 0 \notin V'}$$

reduce to case 1

Chapter 2. projective variety

消去理论 elimination theory

morphisms of varieties.

1. $\mathbb{P}^n = \mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^*$

$= \underbrace{S^{2n+1} / S^1}$ $\quad S^1 \curvearrowright S^{2n+1}$

$\Rightarrow \mathbb{P}^n$ is compact

$\mathbb{P}^n$ is a complex manifold

$$\mathbb{P} = \bigcup_i U_i \quad , U_i := \{ [z_0 : \cdots : z_n] \mid z_i \neq 0 \}$$

$$\varphi_i : U_i \xrightarrow{\sim} \mathbb{C}^n : [z_0, \cdots, z_n] \longmapsto \left(\frac{z_0}{z_i}, \cdots, \frac{z_n}{z_i} \right).$$

φ_i is homeomorphism.

$$\mathbb{P}^n = PGL(n+1, \mathbb{C}) \big/ (PGL(n+1, \mathbb{C}))_p$$

$$PGL(n+1, \mathbb{C}) \times \mathbb{P}^n \longmapsto \mathbb{P}^n$$

$$(A, Z) \longmapsto AZ$$

$$PGL(n+1, \mathbb{C}) = GL(n+1, \mathbb{C}) \big/ \mathbb{C}$$

homogeneous polynomial.

$\Downarrow$

homogeneous ideal.

Projective space.

homogeneous coordinate: $a = [a_0 : a_1 : \cdots : a_n]$

homogeneous ideal:

$$\Leftrightarrow I = \bigoplus_{d \geq 0} (I \cap A_d)$$

Exercise.

(a) I is homogeneous ideal

$\Leftrightarrow$

it is generated by homogeneous elements

(b) I, J homogeneous

$\Rightarrow I+J,\ I\cap J,\ IJ,\ \sqrt{J}$ is homogeneous

(c) A is a graded Noetherian ring, $I \subseteq A$ homogeneous ideal

$\Rightarrow \exists!$ prime decomposition.

$$\sqrt{I} = \bigcap_{i=1}^{r} P_i$$

P_i homogeneous ideal.

$$\bigcap_{i\neq j} P_i \subseteq P_j$$

ity

open sets in algebraic varieties in $\mathbb{P}^n$

is called quasi-projective variety

Hilbert Nullstellsatz

(i) $\pi: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ natural projection

For every $X \subseteq \mathbb{P}^n$

Define $CX := CX = \pi^{-1}(X) \cup \{0\}$

be its cone.

If $X = V(I) \subseteq \mathbb{P}^n$

$\Rightarrow \quad CX = V(I) \subseteq \mathbb{C}^{n+1}$

(ii) For $X \subseteq \mathbb{P}^n$

$I(X) = \{ f \in \mathbb{C}[x_1, \dots, x_n] : f|_X = 0 \}$

$\Rightarrow$ I is a homogeneous radical ideal.

(iii) algebraic preliminaries:

A is a Noetherian ring

$I \subseteq A$ be an ideal

$\Rightarrow \exists N = N(I) \in \mathbb{N}^{*}$ s.t.

$I \supseteq (\sqrt{I})^N$

Projective Nullstellensatz.

$X = V(I) \subseteq \mathbb{P}^n$ be an algebraic subset, $I \subseteq \mathbb{C}[X_0, \cdots, X_n]$ homogeneous ideal,

(i) $X = \phi \iff \sqrt{I} \supseteq (X_0, \cdots, X_n)$

$\overset{(iii)}{\iff} \exists N, \text{s.t. } I \supseteq (X_0, \cdots, X_n)^N$,

i.e. every polynomial of degree $\geq N$ is in I.

(ii) $X \neq \phi \Rightarrow I(X) = I(V(I)) = \sqrt{I}$

proof: (i) TFSAE:

(a). $V(I) = \phi \subseteq \mathbb{P}^n$ (b) $C_X \subseteq \{0\}$

(c) $\sqrt{I} \supseteq (x_0, \cdots, x_n)$ (d) $\exists N, (x_0, \cdots, x_n)^N \subseteq I$

(ii) $I(X) = I(CX)$.

Algebraic Structure of projective variety

(i) $X = V(I) \subseteq \mathbb{P}^n$

$$\gamma_n(X) = \mathbb{C}[x_0, \cdots, x_n] / I(X)$$

be its homogeneous coordinate ring

* Different from affine case, $\gamma_n(X)$ cannot be viewed as functions on

X, $f(\lambda x_0, \cdots, \lambda x_n) = f(x_0, \cdots, x_n)$ is not always true.

$f \in \Gamma_h(X)$, call f a form of degree d, if $\exists F$ of degree d, $f = \bar{F} \mod I(X)$

Zariski topology. (projective space).

closed set $: V(I)$

Affine case:

$$a \longleftrightarrow (a_1, \cdots, a_n)$$

$$\mathbb{C}(X) = \left\{ \frac{f}{g} \in \mathbb{C}(x_1, \cdots, x_n),\ g \notin P \right\} \Big/ \left\{ \frac{f}{g},\ f \in P,\ g \notin P \right\}$$

$= \mathrm{Frac}(\mathcal{Y}(X))$

Local ring $\mathcal{O}_{a,X} \subseteq \mathcal{Y}(X)$

φ regular at a

$\Leftrightarrow \exists \varphi = \frac{f}{g}, \quad g(a) \neq 0$

$\Leftrightarrow \varphi\big|_U = \frac{f}{g}\Big|_U, \quad a \in U, \quad g(b) \neq 0$

for $\forall b \in U$ (germ).

Projective variety.

$\mathbb{C}(X) = \left\{ \frac{f}{g} \in \mathbb{C}[X_0, \cdots, X_r] \,\middle|\, f, g \in S_t, \ g \notin P \right\}$

$$= \left\{ \frac{f}{g} \in \mathbb{C}_h(X) : \exists d \geq 0, f, g \in (\mathcal{S}_h(X))_d \right\}$$

$$\mathcal{S}_h(X) = S/P = \bigoplus_{d \geq 0} (\mathcal{S}_h(X))_d$$

Remark. $\mathcal{S}_h(X) = \bar{S}_d$, require P to be homogeneous.

局部环

$$\mathcal{O}_{a,X} = \left\{ \frac{f}{g} \in \mathbb{C}(X) \;\middle|\; g(a) \neq 0 \right\}.$$

$$\gamma(U, \mathcal{O}_X) = \bigcap_{a \in U} \mathcal{O}_{a,X} \subseteq \mathbb{C}(X).$$

Fact.

$\gamma(X, \mathcal{O}_X) = \mathbb{C}$, i.e.

Regular function over X is const.

拟射影簇

$Y = V(P) \setminus V(I) \subseteq \mathbb{P}^n$, Let $X = V(P)$

Y open in X.

$a \in Y$,

$\mathcal{O}_{a,Y} = \mathcal{O}_{a,X}$

$\mathbb{C}(Y) = \{$ Rational function regular in some open sets of $Y \}$

$= \mathbb{C}(X)$

$$\gamma(u,\mathcal{O}_Y)=\bigcap_{a\in u}\mathcal{O}_{a,Y}=\gamma(u,\mathcal{O}_X)$$

2. maps between algebraic varieties.

Suppose X, Y are two varieties.

A "good" map $\varphi: X\to Y$ should be:

- maps between sets $X\to Y$
- φ is continuous.
- $X\xrightarrow{\varphi}Y\xrightarrow{f}k$

for $\forall\ V \subseteq Y$ open, and $f \in \mathcal{F}(V, \mathcal{O}_Y)$

$$\varphi^* f := f \circ \varphi \in \mathcal{F}(\varphi^{-1}(V), \mathcal{O}_X)$$

Definition.

X, Y varieties.

call $\phi : X \to Y$ a regular map, if

- ϕ is continuous.
- ϕ induced $\phi^* : \mathcal{F}(V, \mathcal{O}_Y) \to \mathcal{F}(\phi^{-1}(V), \mathcal{O}_X)$

$$f \longrightarrow f \circ \varphi$$

Hom(X, Y)

isomorphism: if ϕ, ϕ^{-1} is

morphism.

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i \quad \phi_i : U_i \to \mathbb{C}^n$$

We will prove ϕ_i is isomorphism.

Definition

X is a quasi projective

variety.

If X isomorphism to an

affine variety

call X affine open set

We will prove every quasi projective variety is finite union of affine open sets.

Maps between affine varieties.

$\phi: \mathbb{C}^m \to \mathbb{C}^n$

$(x_1, \cdots, x_m) \mapsto (\phi_1(x_1, \cdots, x_m), \cdots, \phi_n(x_1, \cdots, x_m))$

(1) $\phi^*: \mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(\mathbb{C}^m)$ ring homomorphism.

$$\begin{array}{c} \mathcal{O}(\mathbb{C}^n) \\ \downarrow \wr \\ \mathbb{C}[Y_1, \cdots, Y_n] \to \mathbb{C}[X_1, \cdots, X_m] \end{array}$$

$$\phi^* Y_i = \phi_i(X_1, \cdots, X_m)$$

(2) Conversely, given a ring homomorphism.

$$f: \mathcal{O}(\mathbb{C}^n) = \mathbb{C}[Y_1, \cdots, Y_n] \to \mathbb{C}[X_1, \cdots, X_m]$$
$$\| $$
$$\mathcal{O}(\mathbb{C}^m)$$

Let $\Phi_i = f(Y_i)$

$$\Phi = (\Phi_1, \cdots, \Phi_n): \mathbb{C}^m \to \mathbb{C}^n$$

Fact.

$\varphi = (\varphi_1, \cdots, \varphi_n) : \mathbb{C}^m \to \mathbb{C}^n$ is a polynomial map

$\Rightarrow \varphi$ is regular

- Continuity

$$\varphi^{-1}(V(I)) = \{p \in \mathbb{C}^m \mid f(\varphi(p)) = 0, \forall f \in I\}$$

$$= \{p \in \mathbb{C}^m \mid (f \circ \varphi)(p) = 0, \forall f \in I\}$$

$$= V(\varphi^* f, f \in I) \text{ closed}$$

- $\varphi^*: \mathcal{Y}(V, \mathbb{C}^n) \to \mathcal{K}(\mathbb{C}^m)$

$$\frac{f}{g} \longrightarrow \frac{f}{g} \circ \varphi$$

$$\varphi^*(\mathcal{Y}(V, \mathcal{O}_{\mathbb{C}^n})) = \varphi^*\left(\bigcap_{a \in V} \mathcal{O}_{a, \mathbb{C}^n}\right)$$

$$\subseteq \bigcap_{b \in \varphi^{-1}(V)} \mathcal{O}_{b, \mathbb{C}^m}$$

$$= \mathcal{Y}(\varphi^{-1}(V), \mathcal{O}_{\mathbb{C}^m})$$

Hence we have

$$\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n) \xleftrightarrow{1:1} \mathrm{Hom}(\mathbb{C}[X_1, \dots, X_n],$$
$$\mathbb{C}[X_1, \dots, X_m])$$

More generally,

$$\mathrm{Hom}(X,Y) \overset{1:1}{\longleftrightarrow} \mathrm{Hom}(\gamma(Y), \gamma(X))$$

If X, Y are affine varieties.

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i \qquad \phi_i : U_i \to \mathbb{C}^n$$

Consider ϕ_0.

$$U_0 \longrightarrow \mathbb{C}^n$$

$$\phi_0 : [X_0, \cdots, X_n] \mapsto \left(\frac{X_1}{X_0}, \cdots, \frac{X_n}{X_0}\right)$$

$$\psi_0 : \mathbb{C}^n \longrightarrow U_0$$

$(x_1, \cdots, x_n) \longmapsto [1, \cdots, x_n]$

I will prove ϕ_0, ψ_0 are both regular

$$\deg g = d$$

. Continuous.

$$\psi_0^{-1}(V(g)) = \{[x_0, \cdots, x_n] \in U_0,\ g(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}) = 0\}$$

$$= \{[x_0, \cdots, x_n] \in U_0,\ x_0^d g = 0\}$$

closed.

quasi-affine variety

= open set in affine variety

$= V(P) \setminus V(I),\ P, I \subseteq \mathbb{C}[x_1, \cdots, x_n]$

if X, Y are affine,

$\mathrm{Hom}(X, Y) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(Y), \mathcal{O}(X))$

Definition 1.

X, Y are algebraic varieties

$a \in X$, say $\varphi: X \to Y$ regular at a

if:

(1) φ is continuous at a

(2) $\varphi^* \mathcal{O}_{\varphi(a),Y} \subseteq \underline{\mathcal{O}_{a,X}}$ φ induced maps between local rings.

Lemma. X, Y are two varieties

$\phi : X \to Y$

ϕ is a morphism

$\Leftrightarrow \forall a \in X$, ϕ regular at a

Rational function.

$$\left(\frac{g_1}{h_1}, \cdots, \frac{g_n}{h_n}\right).$$

$\varphi : X \to Y$ regular at $a \in X$

$\Leftrightarrow \varphi = \left(\frac{g_1}{h_1}, \cdots, \frac{g_n}{h_n} \right)$, $h_i(a) \neq 0$.

Projective variety.

Quasi projective variety:

open set in projective variety.

$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$, $\phi_i : U_i \to \mathbb{C}^n$

$\phi_0 : U_0 \to \mathbb{C}^n$ $[X_0, \cdots, X_n] \to \left(\frac{X_1}{X_0}, \cdots, \frac{X_n}{X_0} \right)$

ϕ_0^{-1} : $(\cdot \cdot \cdot)$ $[1, \cdots]$.

(1) ϕ_0 is continuous $\rightarrow$ homogenize

$\phi^{-1}(V(J)) = V(J^*) \cap U_0$

$\varphi_0^{-1}(V(I) \cap U_0) = \{ (x_1, \cdots, x_n) \in \mathbb{C}^n \mid F(1, \cdots, x_n) = 0, F \in V(I) \}$

$= V(I_*)$.

$\rightarrow$ dehomogenize

(2)

$$\varphi_0^* \mathcal{O}_{b,\mathbb{C}^n} \subseteq \mathcal{O}_{a,U_0}$$

$$\forall \frac{f}{g} \in \mathcal{O}_{b,\mathbb{C}^n}$$

$$\varphi^*\left(\frac{f}{g}\right) = \frac{f\left(\frac{X_1}{X_0}, \cdots, \frac{X_n}{X_0}\right)}{g\left(\frac{X_1}{X_0}, \cdots, \frac{X_n}{X_0}\right)}$$

$$= \frac{X_0^d f\left(\frac{X_1}{X_0}, \cdots, \frac{X_n}{X_0}\right)}{X_0^d g\left(\frac{X_1}{X_0}, \cdots, \frac{X_n}{X_0}\right)} \in \mathcal{O}_{a,U_0}.$$

Similarly,

$$\psi^*\left(\frac{F}{G}\right) = \frac{F(1, \cdots, X_n)}{G(1, \cdots, X_n)} \in \mathcal{O}_{b,\mathbb{C}^n}$$

Conclusion.

(1) $\mathbb{C}^n \xrightarrow{\sim} U_0$ 是 quasi - projective

(2) $V(I) = \bigcup_{i=0}^{n} (V(I) \cap U_i)$

- Every quasi-projective set is finite union of quasi affine sets.

Goal: quasi-proj $\Rightarrow$ quasi-affine alg set

$\Rightarrow$ quasi-affine variety

$\boxed{\Rightarrow}$ affine variety.

$X \subseteq \mathbb{C}^n$, $Y \subseteq \mathbb{C}^n$ are two algebraic varieties. $\phi: X \to Y$ is a morphism if and only if ϕ is continuous, an

is regular everywhere

regular at x

$\Leftrightarrow$ rational function.

$$X = V(p) \setminus V(f_1, \cdots, f_r) = \bigcup_{i=1}^{r} (\underbrace{V(p) \setminus V(f_i)}_{\text{variety.}})$$

$D(f) \subseteq \mathbb{C}^n$.

Lemma 1. $D(f) \subseteq \mathbb{C}^n$ is an affine variety. i.e. $D(f)$ is isomorphic to an affine variety.

proof:

$$X = \{(x_1, \cdots, x_n, x_{n+1}) \mid x_{n+1} f(x_1, \cdots, x_n) = 1\}$$

$$\subseteq \mathbb{C}^n$$

$$\varphi: D(f) \longrightarrow X$$

$$(x_1, \cdots, x_n) \longmapsto \left(x_1, \cdots, x_n, \frac{1}{f(x_1, \cdots, x_n)}\right)$$

$\psi : X \longrightarrow D(f)$

$(x_1, \cdots, x_n, x_{n+1}) \mapsto (x_1, \cdots, x_n)$

φ, ψ is rational at everywhere

$\Rightarrow$ they are isomorphisms.

$X = V(x_{n+1} f(x_1, \cdots, x_n) - 1)$

is an affine variety

$\square$

Every quasi-proj variety is a finite union of affine varieties, and every variety is open

In other words, suppose $X \subseteq \mathbb{P}^n$ is a quasi-proj variety

$$X = \bigcup_{i=1}^{n} V_i$$

(1) V_i is iso. to an affine variety

(2) $V_i \subseteq X$ is open.

pf: (1)

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i \qquad \phi_i : U_i \xrightarrow{\sim} \mathbb{C}^n.$$

$$\Rightarrow X = \bigcup_{i=0}^{n} (X \cap U_i)$$

$X \cap U_i$ is an open set of X

$$\phi_i : X \cap U_i \xrightarrow{\sim} V_i$$

$V_i = \phi$ ✓.

if $V_i \neq \phi$

We only need to show

$V_i \xrightarrow{\sim} X \cap U_i$ is irreducible

If $X \cap U_i = X_1 \cup X_2$, where X_1 and X_2 are both closed in $X \cap U_i$

$$X = \overline{X_1} \cup \overline{X_2} \cup (X \cap H_i)$$

$\overline{X_i}$ is the closure of X_i in X.

$X \cap U_i \neq \phi \Rightarrow X \cap H_i \neq X$.

X is irreducible (because $\overline{X}$ is irred.)

$$\begin{array}{c} \overline{X} \\ \cap\!| \\ \mathbb{P}^n. \end{array}$$

$\Rightarrow X \cap U_i = X_1$ or $X \cap U_i = X_2$

$\Rightarrow X \cap U_i = V_i$ is quasi-affine varity

$$V_i = V(\mathfrak{p}) \setminus V(I) = V(\mathfrak{p}) \cap (\mathbb{C}^n \setminus V(f_1, \dots, f_r))$$

$$= \bigcup_{i=1}^{r} \underbrace{V(\mathfrak{p}) \cap D(f_i)}$$

$D(f_i)$ is an affine variety

$V(\mathfrak{p}) \cap D(f_i)$ is open in $V(\mathfrak{p})$

$\Rightarrow$ irreducible

it is closed in $D(f_i)$

$\Rightarrow$ it is an affine variety.

Theorem. Suppose X is a quasi-affine variety,

$\Rightarrow \quad X = \bigcup_{i=1}^{s} U_i$

U_i is open affine variety.

Cor.

$\dim X \leq \dim T_{X,a}$

$\forall a \in X$, $a \in U$, U open variety

$\Rightarrow \mathcal{O}_{a,X} \cong \mathcal{O}_{a,U}$

$\dim X = \text{tr.d.}\ \mathcal{O}_X(X) = \text{tr.d.}\ \mathcal{O}_X(U)$

$= \dim U$

$T_{X,a} = T_{U,a}$.

Product of variety.

$X, Y = \mathbb{C}^n, \mathbb{P}^n, \mathbb{C}^m \times \mathbb{C}^n, \mathbb{C}^m \times \mathbb{P}^n,$

$\mathbb{P}^n \times \mathbb{P}^m$

$f \in P_X$ (polynomials on X).

$D(f) = X \setminus V(f)$. is called by

principle open set.

$\mathbb{P}^m = \bigcup_{i=0}^{m} U_i$, $U_i \xrightarrow[\varphi_i]{\sim} \mathbb{C}^m$.

$$\mathbb{P}^n = \bigcup_{i=0}^{n} V_i, \quad V_i \xrightarrow[\varphi_i]{\sim} \mathbb{C}^n$$

$$\mathbb{P}^m \times \mathbb{P}^n = \bigcup_{i,j} (U_i \cap U_j)$$

$$U_i \times U_j \xrightarrow[\varphi_i \times \varphi_j]{\sim} \mathbb{C}^m \times \mathbb{C}^n \xrightarrow{\sim} \mathbb{C}^{m+n}.$$

(1) Affine coordinate $(X_0, \cdots, \hat{X}_i, \cdots, X_m, Y_0, \cdots, \hat{Y}_j, \cdots)$

(2) Every algebraic variety $X \subseteq \mathcal{X}$ is

$$X = \bigcup_{i=1}^{s} O_i, \quad O_i \subseteq X \text{ open.}$$

---

(1) $X \subseteq \mathcal{X}$ be a algebraic variety.

$Y \subseteq \mathbb{P}$ .....

$\phi: X \to Y$ be any map.

ϕ regular at $a \in X$

$\iff$ (1) continuous at a.

(2) $\phi^* \mathcal{O}_{\phi(a), Y} \subseteq \mathcal{O}_{a,X}$

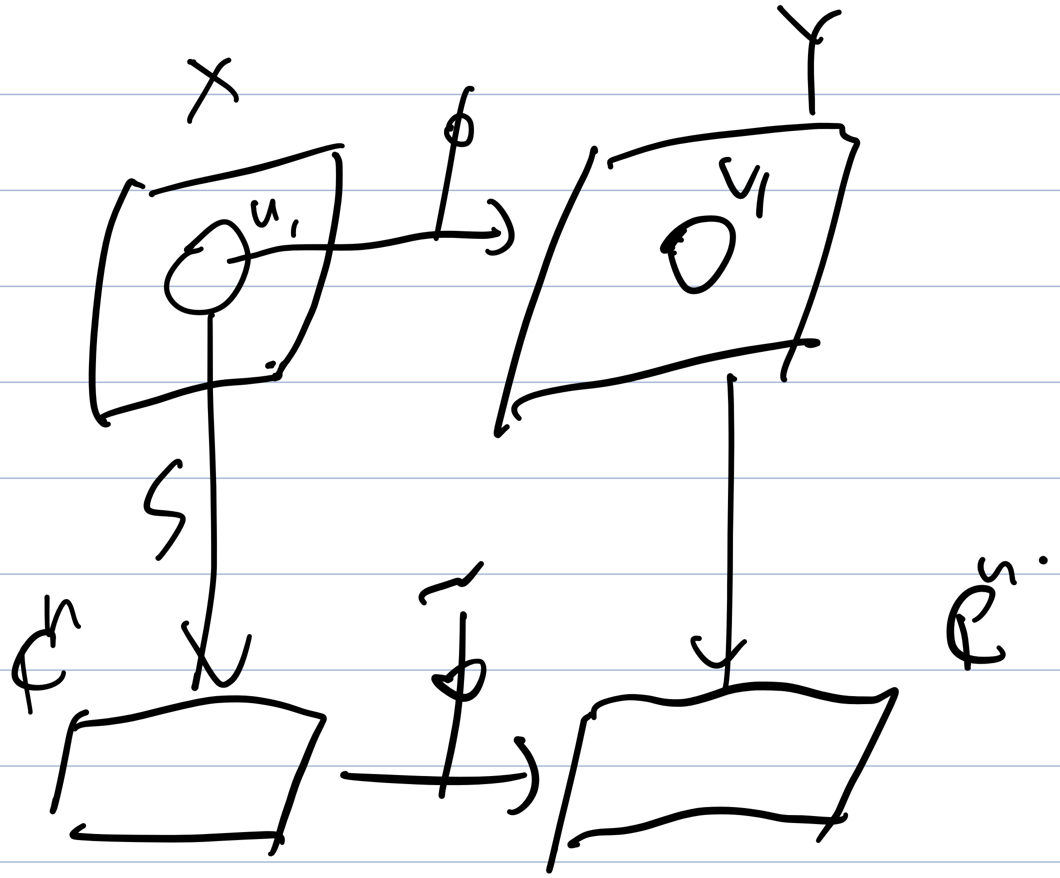
$\iff$ $\forall$ affine open set $\phi(a) \subseteq V_1 \subseteq Y$, $\exists$

affine open set $a \in U_1 \subseteq X$,

$\phi(U_1) \subseteq V_1$

and $\phi|_{U_1}: U_1 \to V_1$ is regular at a

($\Rightarrow$)



$\tilde{\phi}$ is rational at a.

Goal: prove every algebraic variety

in $\mathbb{P}^m \times \mathbb{P}^n$ is algebraic variety.

(a) $\phi: X \to Y$

apply homogeneous coordinate.

$\Rightarrow \phi\big([X_0,\dots,X_m],[Y_0,\dots,Y_n]\big)=[\phi_0,\dots,\phi_N]$

A function $f: X\to\mathbb{C}$ regular at a

$\Leftrightarrow$ (1) $\exists$ neiborhood at $U_{a,f}$.

(2) $\exists$ rational function $\frac{F}{G}$, $G(x)\neq 0$ on $U_{a,f}$,

$f\big|_{U_{a,f}}=\frac{F}{G}\Big|_{U,a,g}.$ $\quad \mathcal{O}_{a,X}=\left\{\frac{F}{G}\,\middle|\,G(a)\neq 0\right\}$

$X\subseteq\mathbb{P}^m\times\mathbb{P}^n\to Y\subseteq\mathbb{P}^N$

$\downarrow$

$a \longmapsto b$

$\mathcal{O}_{a,Y}$ $\quad$ $\mathcal{O}_{b,Y}$

$\Rightarrow \phi$ is rational fuction $\frac{F}{G}$,

F, G are bihomogeneous polynomial of same degree

$X \subseteq \mathbb{P}^m \times \mathbb{P}^n \xrightarrow{\phi} Y \subseteq \mathbb{P}^N$

The morphism is

$\phi = [H_0, \cdots, H_N]$, H_i is bihomogeneous of same degree.

$\Phi : Y \to X$

$\mathbb{P}^N \quad \mathbb{P}^m \times \mathbb{P}^n$

$\boxed{\text{~~五~~ 方}}$

$\Phi([z_0, \cdots, z_n]) = ([\Phi_0, \cdots, \Phi_m], [\Phi_{m+1}, \cdots,$

$\Phi_{m+n+1}])$

$\Phi_0 \sim \Phi_m$ homogeneous of same degree.

$\Phi_{m+1} \sim \Phi_{m+n+1}$.

Segre embedding.

$\mathbb{C}[X_{ij}]_{\substack{0 \le i \le m \\ 0 \le j \le n}}$.

$$\mathbb{P}^m \times \mathbb{P}^n \xrightarrow{\sim} V(X_{ij}X_{kl} - X_{il}X_{kj})_{i,j,k,l}$$

$$(x_i, x_j) \to X_{ij} \qquad \cap\!| \qquad \mathbb{P}^{nm+n+m}.$$

$\mathbb{P}$

$X \subseteq \mathbb{P}^m \times \mathbb{P}^n$

$Y \subseteq \mathbb{P}^n$

$\phi : X \to Y$

$([\underbrace{X_1, \dots, X_m}], [Y_1, \dots, Y_n])$

$[Z_1, \dots, Z_N]$

$a \in D(X_i Y_j)$, $b \in D(Z_k)$

Let $X' = X \cap D(X_i Y_j) \cap \phi^{-1}(D(Z_k))$

reduce to affine case.

$U \otimes V$ $\quad$ U, V are vector spaces.

$$U \times V \longrightarrow U \otimes V$$

$$(U \setminus \{0\}) \times (V \setminus \{0\}) \longrightarrow U \otimes V \setminus \{0\}$$

This induce

projective space.

$$S: \mathbb{P}(U) \times \mathbb{P}(V) \to \mathbb{P}(U \otimes V).$$

Then

$$\begin{array}{c}
\mathbb{P}(\mathbb{C}^{n+1}) \times \mathbb{P}(\mathbb{C}^{m+1}) \\
\downarrow \\
\mathbb{P}(\mathbb{C}^{n+1} \otimes \mathbb{C}^{m+1}) \\
\parallel \\
\mathbb{C}^{nm+n+m+1}.
\end{array}$$

An algebraic variety X is complete.

if $\forall Y$

$$X \times Y \xrightarrow{p} Y$$

is closed map.

Theorem. Projective variety is complete. X projective

$$X \times Y \xrightarrow{p} Y \quad \text{closed.}$$

$$\Updownarrow$$

$$\mathbb{P}^n \times Y \xrightarrow{P} Y \quad \text{closed}$$

Reduce Y to affine variety

$$Y = \bigcup_{k=1}^{r} U_k, \quad U_k \text{ open, affine.}$$

$$\mathbb{P}^n \times Y \xrightarrow{P_Y} Y$$

$$\mathbb{P} \times U_k \xrightarrow{P_{U_k}} U_k$$

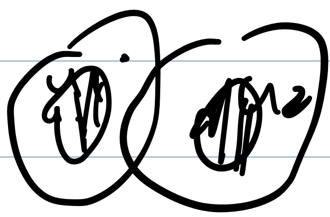
If P_{U_k} are closed

$$P_Y(C) = \bigcup_{k=1}^{r} P_{U_k}(C \cap (\mathbb{P}^n \times U_k))$$

$\forall\, y \in Y \setminus \bigcup_{k=1}^{r} P_{u_k}\left(C \cap \left(\mathbb{P}^n \times U_{(k)}\right)\right)$

$\exists k,\ y \in U_k$

$\Rightarrow \exists\, U,\ y \in U.$



$\Big\Downarrow$ reduce.

$\mathbb{P}^n \times Y \xrightarrow{P} Y$ is projective when Y is an affine variety.

$$\begin{tikzcd} \mathbb{P}^n \times Y \arrow[r, "i"] & \mathbb{P}^n \times \mathbb{C}^m \arrow[d] \\ & \mathbb{P}^n \times Y \arrow[d] \\ & \mathbb{P}^n \times \mathbb{C}^m \end{tikzcd}$$

$$\Downarrow$$

$\mathbb{P}^n \times \mathbb{C}^m \to \mathbb{C}^m$ is closed

Elimination Theorem.

$C = V(f_1, \cdots, f_r)$

$f_i \in \mathbb{C}[X_1, \cdots, X_m, Y_0, \cdots, Y_n]$

f_i is homogeneous for $Y_0 \sim Y_m$.

$$d_i = \deg_{Y_0, \cdots, Y_n} f_i$$

$$f_1(X_1, \cdots, X_m, Y_0, \cdots, Y_n) = 0$$

$$\vdots$$

$$f_r(X_1, \cdots, X_m, Y_0, \cdots, Y_n) = 0$$

Elimination Theory: we can eliminate Y_i, and obtains some polynomial equations of X_i

$$p(C) = \{ a = (a_1, \cdots, a_m) \in \mathbb{C}^m \mid (\mathbb{P}^n \times \{a\}) \cap C \neq \phi \}$$

$$= \{ a \in \mathbb{C}^m \mid f_i(a_1, \cdots, a_m, Y_0, \cdots, Y_n) = 0 \ \forall i \text{ have solution in } \mathbb{P}^n \}.$$

$$\mathbb{C}^m \setminus p(C)$$

$$= \{ a \in \mathbb{C}^m \mid f_i(a_1, \cdots, a_m, Y_0, \cdots, Y_n) = 0 \ \forall i \text{ have no solution in } \mathbb{P}^n \}.$$

$$V(f_1(a, Y), \cdots, f_r(a, Y)) = \phi$$

$$\Leftrightarrow \quad \exists d \geq 0$$

$$(Y_1, \cdots, Y_n)^d \subseteq (f_1(a,Y), \cdots, f_r(a,Y))$$

S_d : all homogeneous polynomoal of degree d.

$$S_d \subseteq \mathbb{C}[Y_0, \cdots, Y_n]$$

$$d_i = \deg f_i$$

$$T^{(d)}(X) : S_{d-d_1} \oplus \cdots \oplus S_{d-d_r} \to S_d$$

$$(h_1, \cdots, h_r) \longmapsto \sum_{i=1}^{r} h_i(Y_1, \cdots, Y_m) f_i(X, Y)$$

$$(Y_0, \cdots, Y_n)^d \subseteq (f_1(a,Y), \cdots, f_r(a,Y))$$

$\Leftrightarrow$ $T^{(d)}(a)$ is surjective

$\Leftrightarrow$ $\operatorname{rank} T^{(d)}(a) \geq \dim S_d$.

$\Leftrightarrow$ $\exists$ $(\dim S_d \times \dim S_d)$ minor of

$T^{(d)}(a) \neq 0$. $\bigcap\limits_{\text{finite intersection.}} \mathcal{O} \Rightarrow$ open.

$\Rightarrow$ open

$\square$

Corollary.

(1) $\phi: X \to Y$, X complete

$\Rightarrow$ $\phi(X)$ is closed, irreducible.

$$\begin{tikzcd} X \arrow[r, "(Id,\phi)"] \arrow[dr, "\phi"'] & X\times Y \arrow[d] \\ & Y \end{tikzcd}$$

Theoren.

If X is a complete variety

$\Rightarrow \gamma(X, \mathcal{O}_X) = \mathbb{C}$

It's enough to prove

$f: X \to \mathbb{C}$ is a constant

function:

$$X \xrightarrow{f} \mathbb{C} \xrightarrow{i} \mathbb{P}^1 \qquad \tilde{f} = i f$$

$$i : z \longmapsto [1 : z]$$

X is complete

$\Rightarrow \tilde{f}(X)$ is a closed subvariety of $\mathbb{P}^1$

$\Rightarrow \tilde{f}(X)$ is a one point set.

(Id, f)

$X \to X \times Y$ is a closed immersion.

$\Rightarrow$ If X is complete

$$\begin{tikzcd} X \arrow[r, "(Id, \phi)"] \arrow[rr, bend right, "\phi"'] & X \times Y \arrow[r, "p"] & Y \end{tikzcd}$$ is closed.

Theorem. If X is a complete variety, X is affine $\Rightarrow$ X is a one pt set.

Pf: $\gamma(X) = \bigcap_{a\in X} \mathcal{O}_{a,X} = \gamma(X, \mathcal{O}_X)$

$\uparrow$ X is affine

$\gamma(X,\mathcal{O}_X) = \mathbb{C}$ (X is complete)

$\Rightarrow$ X is a one pt variety.

Pf 2:

$\exists\ X \overset{i}{\hookrightarrow} \mathbb{C}^N$ closed immersion

x_i can be viewed as regular functions over X

$\Rightarrow \quad x_i \equiv a_i$

$\Rightarrow \ i(X) = \{(a_1, \dots, a_n)\}$

2. Image of morphism.

$\phi: X \to Y$

(1) X is complete $\Rightarrow \phi(X)$ is a closed subvariety

(2) $X = V(x_1 x_2 - 1) \subseteq \mathbb{C}^2$

$$\downarrow$$

$Y = \mathbb{C}^1$

$(x_1, x_2) \to x_2$

is not closed.

Definition.

constructible set (semialgebraic set).

Y is constructible

$\Leftrightarrow \quad Y = T_1 \cup \cdots \cup T_k$

T_i is locally closed

X a quasi-projective set

$T_i \subseteq X$ locally closed

$S = S_1 \cup \cdots \cup S_k$

$S_i \subseteq X$ are subvariety.

Theorem (chevalley theorem)

$\phi: X \to Y$ is a morphism

$\Rightarrow \phi(X)$ is a construtible set

$$S = \bigcup_{i=1}^{r} S_i$$

$$\Rightarrow \bar{S} = \bigcup_{i=1}^{r} \bar{S}_i$$

$\phi: X \to Y$ subvariety

Chevalley Theorem $\Rightarrow \exists\, T_1 \sim T_s \subseteq Y$

$$\Rightarrow \phi(X) = \bigcup_{j=1}^{s} T_j$$

$$\overline{\phi(X)} = \bigcup_{j=1}^{s} \overline{T_j}$$

$\overline{\phi(X)}$ is irreducible

$\Rightarrow \exists\, j_0,\ \overline{\phi(X)} = \overline{T_{j_0}}$

$T_{j_0} \subseteq \phi(X)$ open

$\Rightarrow$ $\phi(X)$ contains a open set of $\overline{\phi(X)}$

Theorem.

X, Y are varieties.

$\phi: X \to Y$ is a morphism

$\Rightarrow \exists$ $U \subseteq \overline{\phi(X)}$ open,

$U \subseteq \phi(X)$

(1) X variety

$S \subseteq X$ constructible set

$\overline{S}^{Z}$ stand for closure of S under Zariski topology.

$\overline{S}^{cl}$ classical topology

$\Rightarrow \overline{S}^{Z} = \overline{S}^{cl}$

[Mumford 74] Cor 1. P. 60

Corollary.

X is a variety

$U \subseteq X$ open

$\Rightarrow \overline{U}^{cl} = X$

Specially, $Sm(X)$ is dense in open

set.

Complete variety = Projective variety.

Theorem 1.

X is a varity, then TFSAE:

(1) X is a projective variety

(2) X is complete

(3). X is compact under classical

topology.

proof: (1) ⟹ (2) have been proved.

(2) ⟹ (3):

$X \hookrightarrow \mathbb{P}^n$ be an embedding.

$X \xrightarrow{\sim} i(X)$

$i(X) \subseteq \mathbb{P}^n$ closed

$\Rightarrow$ $i(X)$ is cmpt.

$\mathbb{P}^n$ is cmpt under classical top.

(3) ⟹ (1)

$$\overline{i(X)}^{cl} = i(X)$$

$$\Rightarrow \overline{i(X)}^{Z} = \overline{i(X)}^{cl} = i(X)$$

$\Rightarrow X \xrightarrow{\sim} i(X) \subseteq \mathbb{P}^N$ is a projective variety

---

$\phi : X \to Y$ X, Y projective

can be expressed as homogeneous polynomial locally.

$$\phi([X_0, \cdots, X_n]) = [P_0, \cdots, P_N].$$

$V(P_0,\cdots,P_N)\cap X=\phi$.

(to ensure $[P_0,\cdots,P_N]$ can be defined).

Linear case:

$(P_0,\cdots,P_N)=(X_0\cdots\cdots X_n)A$

$A\in\mathbb{C}^{N\times n}$

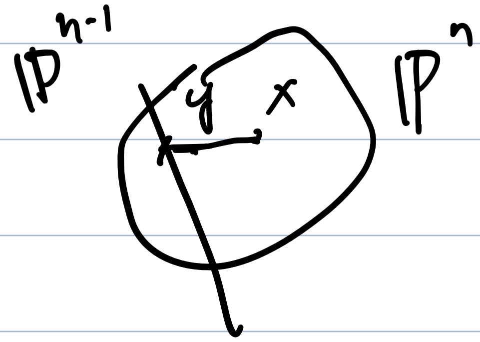
$P_i=X_i$, $N=n-1$

$X_i:=[0,\cdots,0,1]$.

$\pi_x : \mathbb{P}^n \setminus \{x\} \dashrightarrow \mathbb{P}^{n-1}$

$x = [0, \dots, 0, 1]$

Projective with center x.



$y = [y_0, \dots, y_{n-1}, y_n]$ viewed as elements in $\mathbb{C}^{n+1}$.

$y \neq x$ $\quad \ell_y = \{\alpha x + \beta y \in \mathbb{P}^n : [\alpha, \beta] \in \mathbb{P}^1\}$

$\cong \mathbb{P}^1$

$\Leftrightarrow$ $y_0 \sim y_{n-1}$ are not all zero.

To be more generally, if $p_1 \sim p_{n-1}$

linear independent. $\operatorname{rank} A = n-1$

$\Rightarrow \exists B \in GL(n+1, \mathbb{C})$ s.t. $BA = \begin{pmatrix} I_{n-1} \\ 0 \end{pmatrix}$

$\Rightarrow (P_0 \cdots P_{n-1}) = (X_0' \cdots X_{n-1}')$

$[X_0', \cdots, X_n'] = [X_0, \cdots, X_n] \cdot B^{-1}$

$P_i = X_i'$

$V(P_0, \cdots, P_{n-1}) = V(X_0', \cdots, X_n')$ is a out-pt set

---

Projection with center is a linear

subspace.

$N = d$, $P_0 \sim P_d$ linear independent.

Denote $\mathcal{L} = V(P_0, \cdots, P_d)$

$[X_0' \cdots X_n']$

$P_i = X_i'$

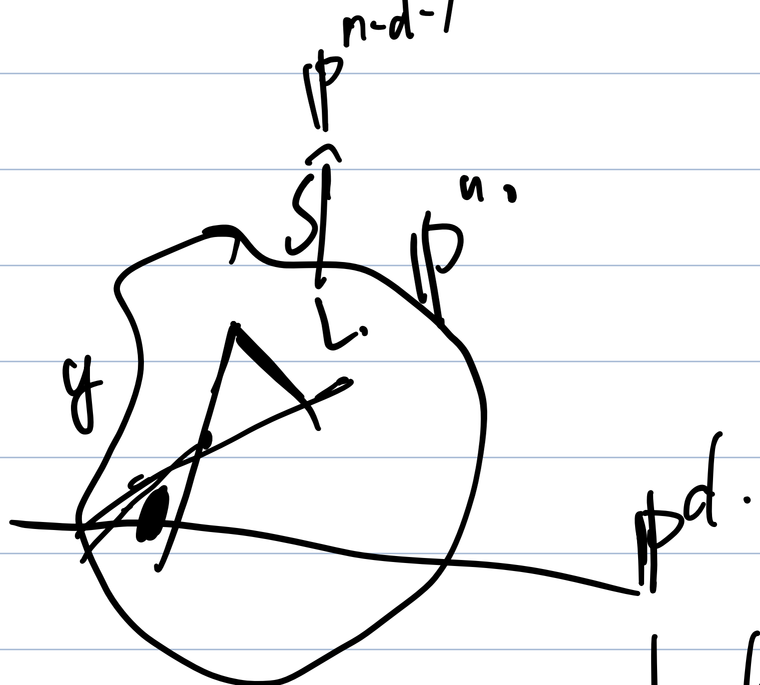
$\mathcal{L} = V(X_0', \cdots, X_d') = \{(0, \cdots, 0, X_{d+1}', \cdots, X_n')\}$

$\subseteq \mathbb{P}^n$.

$\mathcal{L} \xrightarrow{\sim} \mathbb{P}^{n-d-1}$

$\pi_L : \mathbb{P}^n \setminus L \longrightarrow \mathbb{P}^d$

$[X_0', \cdots, X_n'] \to [X_0', \cdots, X_d']$



n-d dimensional.

$L_{,y} = \{$vector space generated by L and $y\}$.

$y = [y_0, \cdots, y_n]$ $\quad L = \{[0, \cdots, 0, X_{d+1}', \cdots, X_n']\}$

$$\overline{L, y} = \{[\alpha y_0, \ldots, \alpha y_d, x'_{d+1}, \ldots, x'_n] \mid [\alpha, x'_{d+1}, \ldots, x'_n] \in \mathbb{P}^{n-d}\}.$$

$X \subseteq \mathbb{P}^n$ projective variety.

$X \subseteq \mathbb{P}^n \setminus L$

$$\pi_L|_X : X \to \mathbb{P}^d.$$

$P_L = \operatorname{im}(\pi_L)$

Noetherian normalization theorem.

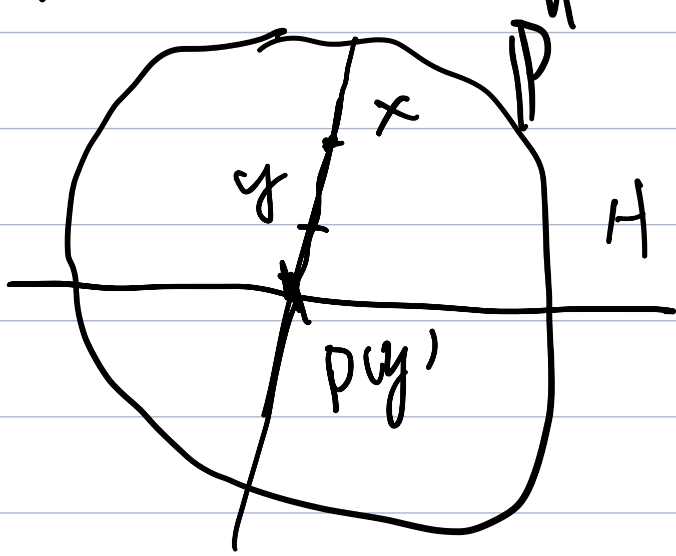
Every d-dimensional projective variety can be viewed as a finite branched cover of $\mathbb{P}^d$

ch：有限分岐覆盖

Suppose $X \subseteq \mathbb{P}^n$ is a d-dimensional projective variety

$$\mathbb{P}^n \setminus X \neq \phi \iff X \neq \mathbb{P}^n$$

$x \in \mathbb{P}^n$



H (hypersurface, $\mathbb{P}^{n-1}$)

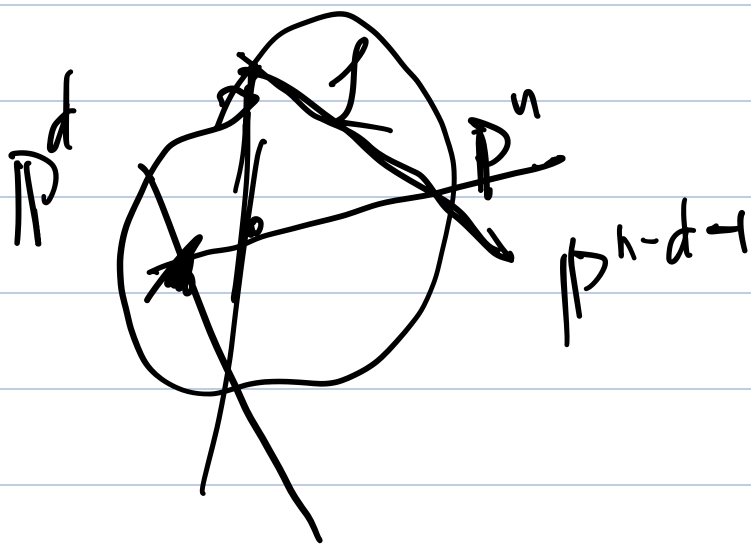
$\mathbb{P}^n \setminus \{x\} \longrightarrow H.$

$\exists\, f : \mathbb{P}^n \xrightarrow{\sim} \mathbb{P}^n$ s.t.

$$\begin{tikzcd}
\mathbb{P}^n \setminus \{x\} \arrow[rr] \arrow[dr, "f"'] & & \mathbb{P}^{n-1} & {[x_0 : \cdots : x_{n-1}]} \\
& \mathbb{P}^n \setminus \{0\} \arrow[ur] & {[x_0 : \cdots : x_n]} \arrow[ur, mapsto] &
\end{tikzcd}$$

$\mathcal{L} \subseteq \mathbb{P}^n$ linear subspace.

$\pi_{\mathcal{L}} : \mathbb{P}^n \setminus \mathcal{L} \to \mathbb{P}^d$



$X \subseteq \mathbb{P}^n$ $\quad X \cap \mathcal{L} = \phi$

$\pi_{\mathcal{L}}\big|_X : X \to \mathbb{P}^d$

Question:

X, Y are projective variety.

$\phi: X \to Y$ is a morphism.

ϕ can locally be given by homogeneous polynomial, globally?

Noetherian Normalization theorem.

$X \subseteq \mathbb{P}^n$ is a n dimensional variety,

then

(1) $\exists\, L \subseteq \mathbb{P}^n$ linear subspace, $L \cong \mathbb{P}^{n-1-d}$

s.t. $L \cap X = \phi$

(2) $\pi_1\big|_X : X \to \mathbb{P}^d$ is a finite morphism

Definition: $\phi: X \to Y$

X, Y projective.

ϕ is a finite morphism, if ϕ is surjective, $\forall y \in Y$, $\phi^{-1}(y)$ is finite

$X^d \subseteq \mathbb{P}^n$ is a projective variety

(X^d means $\dim X = d$)

$d < n \Rightarrow \exists\ x \in \mathbb{P}^n,\ x \notin X^d$ ✓.

Claim: $d < n \Rightarrow X \neq \mathbb{P}^n$

Lemma. Y, Z are two varieties.

$Z \subseteq Y$ is a closed subvariety

$\Rightarrow$ (1) $\dim Z \leq \dim Y$

(2) $\dim Z < \dim Y \Leftrightarrow Y \neq Z$

Pf of Lemma:

(1) Y, Z affine, ✓.

$\forall \quad z \in Z \subseteq Y$

$\exists \overset{z}{\underset{\downarrow}{\ }} U \subseteq Y$, U affine open, s.t.

$$\mathcal{O}_{a,U} = \mathcal{O}_{a,Y}$$

$$\Rightarrow \dim Y = \operatorname{tr.d}_{\mathbb{k}} \mathcal{O}_{z,Y} = \operatorname{tr.d}_{\mathbb{k}} \mathcal{O}_{z,U}$$

$$= \dim U$$

Claim: $Z \cap U$ is an affine open neighborhood of z

$Z \cap U \subseteq U$ is closed.

$Z \cap U$ is irreducible:

Z is irreducible

$$\Downarrow$$

$Z \cap U$ is irreducible.

$Z \cap U \subseteq U$.

$\Rightarrow$ (1) $\dim Z \cap U \leq \dim U$

$$\begin{array}{ccc} \dim Z\cap U & \leq & \dim U \\ \| & & \| \\ \dim Z & & \dim U \end{array}$$

(2) if $\dim Z = \dim U$

$\Rightarrow \ Z \cap U = U$

$$\Rightarrow \begin{array}{ccc} \overline{Z \cap U} & = & \bar{U} \\ \text{\rotatebox{90}{$\subseteq$}} & & \| \\ Z & & Y \end{array}$$

$\square$

$\Rightarrow$ The claim is true.

If $d = n$, $X = \mathbb{P}^n$ ✓

If $d < n \Rightarrow \exists x \in \mathbb{P}^n \setminus X$

$\pi_x : \mathbb{P}^n \setminus \{x\} \to \mathbb{P}^{n-1}$

induced $P_x : X \to \mathbb{P}^{n-1}$

Theorem 1:

$X \subseteq P^n$ is a d dimensional projective variety

$p = P_x : X \to \mathbb{P}^{n-1}$

Then:

(1) $P(X)$ is a subvariety of $\mathbb{P}^{n-1}$

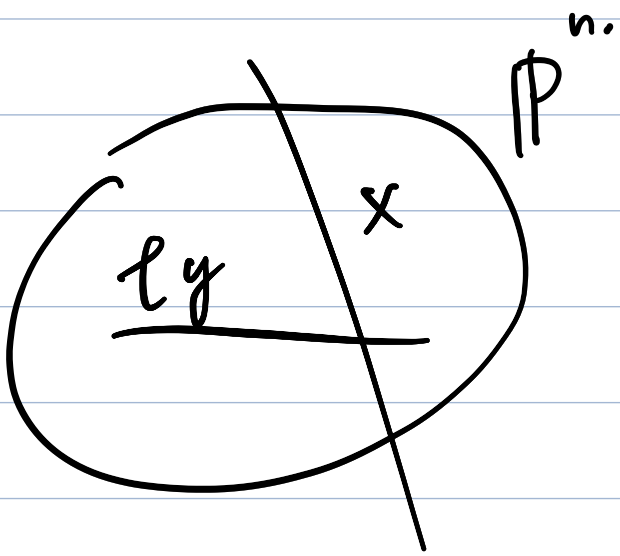
(2) $X' = P(X)$, $\dim X' = \dim X = d$.

(3) $p : X \to X'$ is a finite morphism.

Proof: (1): ✓

(3): $\forall y \in X'$

$p^{-1}(y) = \ell_y \cap X$



$\ell_y \cap X \subsetneq \ell_y \cong \mathbb{P}^1$

$\Rightarrow$ $\ell_y \cap X$ is a proper closed set of $\mathbb{P}^1$, hence is finite.

(2): step I.

$$\pi^* : \mathbb{C}[Y_0, \cdots, Y_{n-1}] \to \mathbb{C}[X_0, \cdots, X_n]$$

$$Y_i \longrightarrow X_i$$

$$\pi^* : \mathbb{C}[Y_0 \cdots Y_{n-1}] \to \mathbb{C}[X_0, \cdots, X_n] \to \mathbb{C}[X_0 \cdots X_n] / I(X) = \gamma_h(X)$$

$$\boxed{\ker \pi^* = I(X')}$$ check -

$$\Downarrow$$

$$p^* : \gamma_h(X') \longrightarrow \gamma_h(X)$$

$$\gamma_h(X') = \mathbb{C}[Y_0, \cdots, Y_{n-1}] / I(X') \qquad \gamma_h(X) = \mathbb{C}[X_0, \cdots, X_n] / I(X)$$

p^* is injective.

$\mathcal{Y}_n(X)$ is a $\mathcal{Y}_n(X')$ algebra

generated by $\bar{X}_n$.

Claim: $\text{tr.d. } \mathbb{C}(X) = \text{tr.d. } \mathbb{C}(X')$

$$\begin{array}{ccc} \text{tr.d. } \mathbb{C}(X) & = & \text{tr.d. } \mathbb{C}(X') \\ \| & & \| \\ \dim X+1 & & \dim X'+1. \end{array}$$

$\Leftarrow$ $\bar{X}_n$ is algebraic over $\mathbb{C}(X')$

$\Leftarrow$ $\exists F$ homogeneous,

$F(X_0, \cdots, X_n) = X_n^d + \square X_n^{d-1} + \cdots$

$\theta = \overline{F(X_0, \cdots, X_n)} \in \mathcal{J}_h(X)$

If $X \cap D(X_n) = \phi$, let $F = X_n$.

If $X \cap D(X_n) \neq \phi$.

Consider in $D(X_n)$.

$\square$

Theorem,

$X \subseteq \mathbb{P}^n \quad \dim X = d$

$\Rightarrow \exists$ $(n-d-1)$ dimensional linear subspace L s.t. $L \cap X = \emptyset$

$p_L : X \to \mathbb{P}^d$ is surjective finite morphism.

$\phi : X \to Y$ is a morphism.

$\phi^{-1}(y)$?

$\phi : X^d \to \mathbb{P}^d$

$\phi^{-1}(y)$?

Theorem 1: $\phi: X \to Y$ is a

surjective morphism

$\Rightarrow$ $\phi^*: \gamma(Y, \mathcal{O}_Y) \to \gamma(X, \mathcal{O}_X)$

is injective.

Specially:

(1) $\phi^*: \gamma(V, \mathcal{O}_Y) \to \gamma(\phi^{-1}(V), \mathcal{O}_X)$

is injective

(2) $\dim X \geq \dim Y$.

Pf: $\phi^* f = 0$

$\Rightarrow$ $f \circ \phi = 0$

$\Rightarrow$ $f = 0$.

2) : Choose $V \subseteq Y$ affine open.

$U \subseteq \phi^{-1}(V)$ affine open

$\gamma(V, \mathcal{O}_Y) \hookrightarrow \gamma(\phi^{-1}(V), \mathcal{O}_X) \hookrightarrow \gamma(U, \mathcal{O}_X)$

$\Rightarrow$ This induce

$$\mathbb{C}(V) \hookrightarrow \mathbb{C}(U)$$

$\square$

Dominant morphism (支配态射)

ϕ is dominant if

$$\overline{\phi(X)} = Y$$

Theorem 2:

$\phi : X \to Y$ is dominant.

$\Rightarrow \phi$ induce $\mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$

proof: $\forall V \subseteq Y$, $\phi^{-1}(V) \neq \emptyset$.

$$\gamma(V, \mathcal{O}_Y) \to \gamma(\phi^{-1}(V), \mathcal{O}_X)$$

morphism of direct system.

$$\begin{tikzcd}
\gamma(V, \mathcal{O}_Y) \arrow[r, "\phi^*"] \arrow[d] \arrow[dr] & \gamma(\phi^{-1}(V), \mathcal{O}_X) \arrow[d] \\
\mathbb{C}(Y) \arrow[r, dashed, "\exists !"] & \mathbb{C}(X)
\end{tikzcd}$$

Chevalley theorem

$\Rightarrow$ If ϕ is dominant

$\exists\, U$ open, $U \subseteq \phi(X)$

3. Local property.

$\phi: X \to Y$

$\phi(x) = y$

$\forall\, V \ni y$ affine open

$x \in U \subseteq \phi^{-1}(V)$ affine open

$\Rightarrow \quad \phi : U \to V$

$X^r \subseteq \mathbb{C}^n \quad Y^s \subseteq \mathbb{C}^m$

affine

$\Rightarrow$ (1) $\phi = (\phi_1, \cdots, \phi_m)$

(2) $\phi^* : \mathcal{O}(Y^s) \to \mathcal{O}(X^r)$

$Y_i \mapsto \phi_i(X_1, \cdots, X_n)$

If ϕ is dominant

$\Rightarrow \phi^*$ is injective.

$\left(\begin{array}{l} \text{If } f\circ\phi = 0 \Rightarrow \text{Im}\,\phi \subseteq V(f) \\ \quad \Rightarrow f = 0 \end{array} \right)$

(3) map of tangent space

$\phi(x) = y$

$\phi^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$

$\phi^*(m_y) \subseteq m_x$

$\left(\phi^* f(x) = f(y) \right)$

$\Rightarrow \phi^*(m_y^2) \subseteq m_x^2$

$\Rightarrow \phi^*$ induces

$m_y / m_y^2 \to m_x / m_x^2$

$$\Rightarrow \quad (m_x/m_x^2)^* \to (m_y/m_y^2)^*$$

$$\begin{array}{c} \| \\ \mathrm{Hom}(m_x/m_x^2, \mathbb{C}) \end{array}$$

ϕ ,

Recall.

$$T_{x,X} = \left\{ (\xi_1, \dots, \xi_n) \;\middle|\; \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \xi_i = 0, \quad \forall f \in I \right\}$$

$$(d\phi)_x(\xi_1, \dots, \xi_n) = \left(\sum_{i=1}^{n} \frac{\partial \phi_1}{\partial x_i} \xi_i, \dots, \sum_{i=1}^{n} \frac{\partial \phi_m}{\partial x_i} \xi_i \right)$$

$X^r, Y^s, \quad \phi: X^r \to Y^s$

Definition.

$x \in X, \ y = \phi(x) \in Sm(Y)$

If: (1) x is smooth

(2) $(d\phi)_x: T_{x,X} \to T_{y,Y}$

is surjective

$\Rightarrow \phi$ is smooth at x

Remark. y can be singular.

$$\mathrm{Sm}(\phi) = \{x \in X \mid \phi \text{ smooth at } x, \; y = \phi(x) \in \mathrm{Sm}(Y)\}$$

$$\Rightarrow \dim T_{x,X} = r \qquad \dim T_{y,Y} = s$$

$$\dim \ker (d\phi)_x = r - s$$

Theorem (generic smoothness)

X^r, Y^s

$\phi : X \to Y$ is dominant

$\Rightarrow \quad \Sigma_m(\phi)$ is a non-empty open set.

pf: $\quad X = V(f_1, \cdots, f_k) \quad f_i \in \mathbb{C}[x_1, \cdots, x_n]$

$\phi = (\phi_1, \cdots, \phi_m), \ \phi_i \in \mathbb{C}[x_1, \cdots, x_n]$

Let

$$M = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \\ \hline \end{bmatrix}$$

$$\begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial \phi_m}{\partial x_1} & \cdots & \frac{\partial \phi_m}{\partial x_n} \end{pmatrix}$$

$$= \begin{pmatrix} A \\ \\ B \end{pmatrix}$$

$$\tilde{A}(x) : \mathbb{C}^n \to \mathbb{C}^k$$

$$v \to A(x) \cdot v$$

$$\tilde{B}(x) : \mathbb{C}^n \to \mathbb{C}^m$$

step 1:

prove: (1) $x \in X - \phi^{-1}(\mathrm{Sing}(Y))$

$\Rightarrow \mathrm{rank}(M(x)) \leq n-r+s$

(2) $x \in \mathrm{Sm}(\phi)$

$\Leftrightarrow \mathrm{rank}(M(x)) = n-r+s$

(1) $T_{x,X} = \ker \tilde{A}(x) \Rightarrow \mathrm{rank}\, A \leq n-r$

$\mathrm{rank}\, A(x) = n-r \Leftrightarrow x \in \mathrm{Sm}(X)$

$\tilde{B}(x) : T_{x,X} \to T_{y,Y}$?

$\Rightarrow$ $\operatorname{rank}(B(x)) \leq s$

(2) $x \in Sm(\varphi)$

$\Leftrightarrow$ $\operatorname{rank} A = n-r$

$\operatorname{rank} B = s$

X, Y be two variety

$\phi : X \to Y$

$\phi : X \to \overline{\phi(X)}$ is dominant.

$T_{x,X} \to T_{y,Y}$

$T_{x,X} = \{ (a_1, \cdots, a_n) \mid \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} a_i, \forall f \in I(X) \}$

$\phi : X \to Y$

$y \in Sm(Y)$

ϕ is smooth at x, if

(1) $x \in Sm(X)$

(2) $(d\phi)_x : T_{x,X} \to T_{y,Y}$ is surjective

Theorem (generic smoothness).

$M(x)$:

$x \in Sm(\phi)$

$\Leftrightarrow$ rank $M(x) = n+t+s$

step 2.

$$\tilde{M} : (\mathbb{C}(X))^{n} \to (\mathbb{C}(X))^{k+m}$$

$$v \longrightarrow Mv$$

step 3. $g_1, \cdots, g_t$ is all

$(n-r+s)$ minors of $M(x)$

$\Rightarrow U = X - V(g_1, \cdots, g_k) - \phi^{-1}(\mathrm{Sing}(Y))$

non empty.

$\square$

Cor. $\mathrm{Sm}(\phi)$ is a submanifold of $\mathbb{C}^n$

Theorem 2. (Sard theorem)

X, Y are affine variety,

$\phi: X \to Y$ is dominant

$\Rightarrow \exists\ Y_0 \subseteq Y$ non-empty, open

s.t. $\phi^{-1}(Y_0) - \mathrm{Sing}(X) \subseteq \mathrm{Sm}(\phi)$

pf: let $X_0 = \mathrm{Sm}(\phi)$

$W = X - X_0 - \mathrm{Sing}(X) - \phi^{-1}(\mathrm{Sing}(Y))$

is locally closed.

- $Y_0 = Y - \overline{\phi(W)} - \mathrm{Sing}(Y)$

It will be suffice to prove

$\overline{\phi(W)} \neq Y$

If $\overline{\phi(W)} = Y$

$\Rightarrow$ $W_1 \xrightarrow{\phi|_{W_1}} Y$ is dominant

(W_1 is an irreducible component of W)

$\Rightarrow$ W_1 contains smooth point,

x

$x \in Sm(\phi)$ $\quad X^r \xrightarrow{\phi} Y^s$

$\Leftrightarrow \dim \ker d\phi = r-s$

$\Leftrightarrow \dim \ker M(x) = r-s$

$\Leftrightarrow \operatorname{rank} M(x) = n-r+s.$

$$M(x) = \begin{pmatrix} \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j} \\ \left(\frac{\partial \phi_i}{\partial x_j}\right)_{i,j} \end{pmatrix} \qquad \square$$

Cor. X, Y are affine varieties

$\phi : X \to Y$ is dominant, then

(1) $\exists\, Y_0 \subseteq Y$ open, s.t.

$\phi^{-1}(Y_0) = X_0 \cup \cdots \cup X_K$

- X_i is locally closed
- $X_i \cap X_j = \phi$
- X_i is smooth.

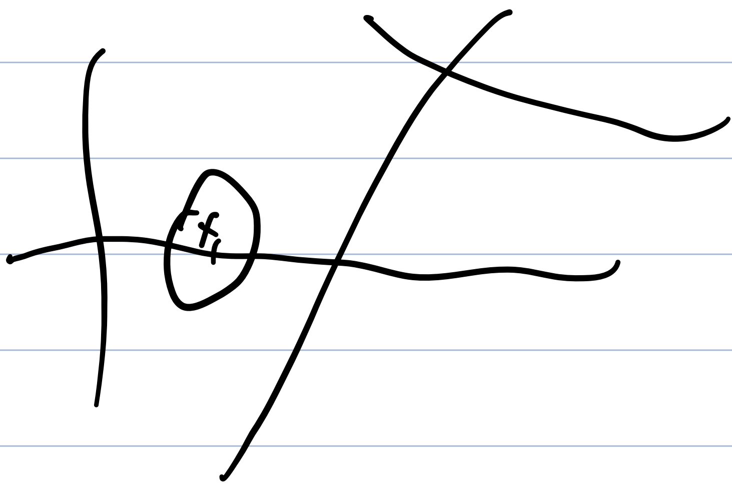
(2) $\phi|_{X_i} : X_i \to X_0$ is smooth

(3) $\dim X_0 = X$, $\dim X_i < \dim X_j$, $\forall\, i > j$

基本开性原理.

Fundamental openness principle

(1) Definition 1. X^r is affine



$x \in X^r$ has topologically unibranch,

$\{U_n\}_n$ 邻域基.

if $\exists\ \lambda \in U_n$ (classical topology)

$x \in U$, $U \cap X^{a}$ is connected

(classical topology)

e.g. If $x \in \mathrm{Sm}(X)$

$\Rightarrow$ X is unibranch.

Whitney decomposition.

$\phi : X \to Y$ $\qquad \phi(X_0) = Y_0 \subseteq Y$ open

$\phi^{-1}(y) \cap X_0$

$r = \dim X \qquad s = \dim Y$

$\phi^{-1}(y) \cap X_0$ is a subcomplex

manifold.

$\Rightarrow \dim(f^{-1}(y) \cap X_0) \leq r-s$

Fundamental openness principal.

Unibranch.

$x \in Sm(X) \Rightarrow \exists\ x \in U \subseteq X$, s.t.

$\exists$ holomorphic map

$\varphi: U \xrightarrow{\sim} B_1 \subseteq \mathbb{C}^r$

$x \longrightarrow 0$

$\Rightarrow B_1 - \varphi(U \cap Y)$ is connected.

定理27.1. （基本开性原理）设X^r, Y^r是两个r维仿射簇，$\phi: X \to Y$是一个支配态射。设$x \in X, y = \phi(x) \in Y$满足

(a) $y = \phi(x)$是Y的*unibranch*点；

(b) $\{x\}$是$\phi^{-1}(y)$的一个不可约分支（等价地，连通分支，孤立点）。

则在经典拓扑下，ϕ在x处是开映射，*i.e.*, $\forall x$在X中的邻域U，$\exists y$在Y中的邻域V s.t. $V \subset \phi(U)$。

This is to say ϕ is a branched cover at x.

$$\mathrm{mult}_x(\phi) := \#(\phi^{-1}(y) \cap U)$$

☆. Dimension of fiber

Suppose $\phi: X^r \to Y^s$ is dominant.

$\exists Y_0 \subseteq Y$ is a non-empty open set, s.t. $\underline{\forall y \in Y, \dim \phi^{-1}(y) \leq r-s}$

[This have been proved by complex manifold]

Theorem. X^r, Y^s are affine varieties

$\phi: X \to Y$ is dominant

$\Rightarrow \forall y \in \phi(X)$, $\forall$ irreducible component W of $\phi^{-1}(y)$

$\dim W \geq r-s$

Pf: if $\exists y \in \phi(X)$

s.t. $\phi^{-1}(y) = \bigcup_{i=1}^{k} W_i$ is the

irreducible decomposition of $\phi^{-1}(y)$

s.t. $W = W_i$, $\dim W < r-s$

Step 1: Choose $x \in W \setminus \bigcup_{j \neq i} W_j$

$y = \phi(x)$

Lemma 1: Suppose $S \subseteq \mathbb{C}^n$ is a closed affine set, $p \in S$, $s = \dim S$

$\Rightarrow \exists\, g_1, \dots, g_s \in \mathbb{C}[x_1, \dots, x_n]$

s.t. $\{p\}$ is a component

of $S \cap V(g_1, \dots, g_n)$

Pf: use induction.

$s = 0$, ✓.

$s > 1$: ...... ✓.

from Lemma 1.

$\exists s$ polynomials $g_1 \sim g_s$ s.t.

$\{y\}$ is a component of

$Y \cap \{g_1, \cdots, g_s\}$

For $\{x\} \subseteq W$, $\dim W < r-s$

$\exists f_1 \sim f_{r-s-1}$, s.t.

$\{x\}$ is a component of

$w \cap V(f_1, \cdots, f_{r+s-1})$

Step 2.

$$\psi : X \to \mathbb{C}^{r-1}$$

$$z \to \big(f_1(z), \cdots, f_{r-s-1}(z),$$

$$g_1(\varphi(z)), \cdots, g_s(\varphi(z))\big)$$

$z \in \psi^{-1}(0)$

$\Longleftrightarrow z \in X \cap V(f_1, \cdots, f_{r-s-1})$

$\varphi(z) \in Y \cap V(g_1, \cdots, g_s)$

$\Rightarrow \{x\}$ is an irreducible component of $\psi^{-1}(0)$

If $x \in W'$ is an irreducible component of $\psi^{-1}(0)$, then

$y \in \overline{\phi(W')} \subseteq Y \cap V(g_1, \cdots, g_s)$

$\Rightarrow \overline{\phi(W')} = \{y\}$.

$\Rightarrow W' \subseteq \phi^{-1}(y)$

$\Rightarrow W' = \{x\}$

Step 3.

$k := \min\{\ell : \exists$ morphism $\Psi : X \to \mathbb{C}^{\ell}$, s.t. $\{x\}$ is an irre. component of

$f^{-1}(0)\}$

(1) Step 2. $k \leq r-1$

(2) $\exists\ \varphi : X \to \mathbb{C}^k$, s.t. $\{x\}$ is a component of $\varphi^{-1}(0)$

Claim: $\varphi: X \to \mathbb{C}^k$ is dominant

or $\dim \overline{f(X)} = k' < k$

Consider morphism:

$$H\circ\psi : X \to \mathbb{C}^{k'}$$

$$H : \mathbb{C}^{k} \to \mathbb{C}^{k'}$$

$H=(h_1,\dots,h_k) \Rightarrow \{0\}$ is a component of $(H\circ\psi)^{-1}(0)$.

[Noether Normalisation?)

$\times$.

Step 4.

$\psi : X^r \to \mathbb{C}^k$, $k \le r-1$

$Y_1 \sim Y_k \in \mathbb{C}[Y_1, \cdots, Y_k]$

$\Rightarrow \varphi^* Y_1, \cdots, \varphi^* Y_k$ are algebraically independent.

$\varphi^* Y_1, \cdots, \varphi^* Y_k, P_{k+1}, \cdots, P_r \in \mathbb{C}(X)$

is alg independent

$$\tilde{\varphi}: X \to \mathbb{C}^r = \mathbb{C}^k \times \mathbb{C}^{r-k}$$

$$z \to (\varphi(z), P_{k+1}(z), \cdots, P_r(z))$$

$\Rightarrow \cdot\ \tilde{\varphi}$ is dominant

If $\overline{\tilde{\varphi}(X)} \neq \mathbb{C}^r$, $\exists\, 0 \neq F$, s.t.

$\left(f|_{\overline{\varphi(X)}} = 0 \Rightarrow f(\varphi^* t_1, \dots, \varphi^* t_r, p_{r+1}, \dots, p_r) = 0 \right)$

......

$X^r \xrightarrow{\phi} Y^r$ dominant

$\Rightarrow \exists\, Y_0 \subseteq Y$ open, s.t.

$\# \phi^{-1}(a) = [\mathbb{C}(X) : \mathbb{C}(Y)]$

degree of field extension

$\forall a \in Y_0$.

$\deg \phi = [\mathbb{C}(X) : \mathbb{C}(Y)]$

(2) Suppose X^r, Y^r are projective

$\phi : X \to Y$ is finite

$y \in Y$ is unibranch

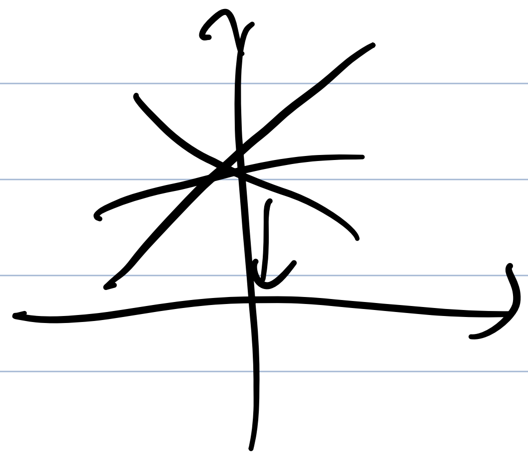
$\Rightarrow \sum_{x \in \phi^{-1}(y)} \text{mult}_x(\phi) = \deg \phi$

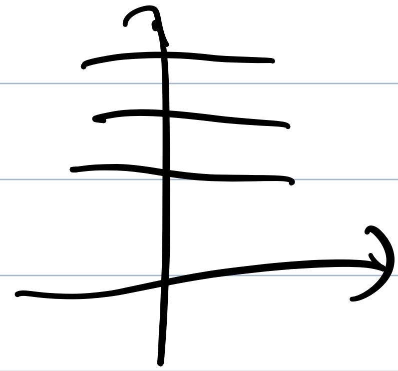
$= [\mathbb{C}(X) : \mathbb{C}(Y)]$

$\Rightarrow \# \phi^{-1}(y) \leq \deg \phi$

$X^d \subseteq \mathbb{P}^n$ is projective /

$\phi: X^d \to \mathbb{P}^d$ is finite





半连续性 (Semi continuity)

$\phi: X^r \to Y^s$

$f(x) = \dim \phi^{-1}(\phi(x))$

is upper-continuous

i.e. $\{x \in X,\ \dim \phi^{-1}(\phi(x)) \geq k\}$

is Zariski closed.

Dimension. tangent space.

$V(f_1, \cdots, f_r)$

Holomorphic function.

Algebraic set $\Rightarrow$ Analytic set.

(1) $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ 的解析函数

为常数!

(2) Riemann surfaces 为代数曲线

Chow's theorem:

解析簇都是代数簇.

$\mathbb{P}^n$ 中任意闭的解析子集(由全纯函数给出)都是 $\mathbb{P}^n$ 中的射影集.

解析子集(analytic subset).

定义. $U \subseteq \mathbb{C}^n$ 为一开集(经典拓扑)

$X \subseteq U$ 为 U 中一闭集

称 X 为 U 的解析子集，若

(1) $\forall\, x \in X$，$\exists\, x \in U$ 的开邻域 $U' \subseteq U$，

s.t.

(2) $\exists$ 有限多解析函数 $\{f_1, \dots, f_r\}$

$$X \cap U' = \{x \in U' : f_i(x) = 0\}$$

解析子集：局部由解析函数给

$\mathcal{O}_{\{x_1, \dots, x_n\}}$

$= \{$收敛的幂级数 $\sum c_\alpha x^\alpha\}$

$= \mathbb{C}\{x_1, \cdots, x_n\}$ 为 Noether 环.

零点定理.

Rückert's Nullstellensatz.

不可约分解.

$X \subseteq U$ 为解析子集

称 X 不可约, 若

(1) $X \neq \phi$

(2) 若 $X = X_1 \cup X_2$, X_1, X_2 为 X 中

闭集, 且为 0 的邻域 U' 中解析集

那么 $X = X_1$ 或 X_2

任意解析集有不可约分解.

定理 1.

任意解析集存在唯一不可约分

解. $X = \bigcup_{i \in I} X_i$

(3) Dimension, smooth.

$X \subseteq U \subseteq \mathbb{C}^n$ 为一解析子集, 称

X 为 $x \in U$ 处的一个解析流形, 若

$\exists u' \ni x,\quad X \cap u' = V(f_1, \cdots, f_k)$

$$\operatorname{rank} \frac{\partial(f_1, \cdots, f_k)}{\partial(x_1, \cdots, x_n)} = k$$

定理 2.

$$X = Sm(X) \sqcup Sing(X)$$

其中 $Sm(X)$ 为稠密开集，r维复

流形

$\dim X := r$

$Sing(X)$ 的任意不可约分支的维数

$< r$

*-解析子集

设 $U \subseteq \mathbb{C}^n$ 为一经典拓扑下开集

称 X 为 U 中的一个*解析子集，

若有以下分解：

$$X = X^{(r)} \cup \cdots \cup X^{(0)}$$

其中 · $X^{(i)}$ 为 i 维子流形

· $\forall i \le r$ $\quad \overline{X^{(i)}} \subseteq X^{(i)} \cup \cdots \cup X^{(0)}$

任一代数集可分解为子流形之并.

定理4. 任意解析集均为 * - 解析集

Remert-Stein 定理.

设 $U \subseteq \mathbb{C}^n$ 中为一开集，$F \subseteq U$ 为闭解析子集，$X \subseteq U \setminus F$ 为一解析子集，满足

设 $X = \bigcup_{i \in I} X_i$，$F = \bigcup_{j \in J} F_j$ 为不可约分解.

$\forall x \in X$

- $\inf_{x \in X_i} \dim X_i > \sup_{j \in J} \dim F_j$

则 X 在 U 中闭包为 U 中解析子集

因此，对解析集

$$X = X^{(r)} \cup \cdots \cup X^{(j)}$$

$$X^{(r)} \subseteq U \setminus (X^{(r-1)} \cup \cdots \cup X^{(j)})$$

$\Rightarrow \overline{X^{(r)}}$ 为 U 中解析集

同理 $\overline{X^{(i)}}$ 为 U 中解析集

$\Rightarrow X = \overline{X^{(r)}} \cup \cdots \cup \overline{X^{(j)}}$ 为解析集

Chow's theorem.

推论1：设 $X \subseteq \mathbb{P}^n$ 为闭解析子集，则 X 为 $\mathbb{P}^n$ 中射影集

Pf：step 1.

$\pi: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ 为自然投影.

$$CX = \pi^{-1}(X) \cup \{0\}$$

X 解析 $\Leftrightarrow$ CX 解析.

$$X = X^{(k)} \cup \cdots \cup X^{(0)}$$

$$\Rightarrow CX = CX^{(k)} \cup \cdots \cup CX^{(0)}$$

$\Rightarrow$ CX 为解析集.

Step 2.

$\exists \delta > 0$, s.t. $B_\delta(0) = B_\delta$

$\exists f_1 \sim f_k$ 全纯, s.t.

$$CX \cap B_\delta = \{ p \in U \mid f_1(p) = \cdots = f_k(p) = 0 \}$$

Step 3.

记 $f_i(x) = \sum C_\alpha X^\alpha = \sum_{r \geq 0} f_{i,r}(x)$

$$f_{i,r} = \sum_{|\alpha| = r} C_\alpha X^\alpha$$

固定 $x \in CX \cap B_\delta$

$$f_i(\lambda x) = \sum_{r\geq 0} \lambda^r f_{i,r}(x)$$

$|\lambda| \leq 1$

$\Rightarrow f_{i,r}(x)=0,\ \forall i,r$

$\Rightarrow CX \cap B_\delta \subseteq V(f_{i,r},\ i\leq k,\ i\geq 0)$

由于 CX, $V(f_{i,r},\ i\leq k)$ 均为锥

$\Rightarrow CX = V(f_{i,r},\ i\leq k)$

$\Rightarrow X$ 为射影集.

奇异簇 $\xrightarrow{?}$ 光滑簇.

给定一个代数簇，能否自持 Sm(X)，将X变为光滑簇.

双有理等价

消除有理映射的不定点

Rational map:

X, Y are varieties.

X到Y的有理映射是指X的某非空开集到Y的一个态射.

变正式的定义.

X到Y的有理映射指的是

X的非空开集到Y的等价类

(i) $U \subseteq X$ 开集, $\varphi_U: U \to Y$ 态射

记为 $\langle U, \varphi_U \rangle$

$\langle U, \varphi_U \rangle \sim \langle V, \varphi_V \rangle$, 若 $\exists W \subseteq U \cap V$ 开

$$\varphi_U\big|_W = \varphi_V\big|_W$$

(ii) 一个有理映射 $\varphi: X \to Y$

指(i)中一个等价类，即 $\exists\ \varphi_u$, s.t.

$\phi = [\langle u, \varphi_u \rangle]$

此时 $\phi(p) = \varphi_u(p)$, $\forall p \in u$ 良定义.
?

(ii) 给定有理映射 $\phi: X \dashrightarrow Y$, $p \in X$

称 ϕ 在 p 处有定义，若 $\exists \langle u, \varphi_u \rangle$, s.t.

$\phi = [\langle u, \varphi_u \rangle]$, $p \in u$

记 $\mathrm{Dom}(\phi) = \{p \in X \mid \phi$ 在 p 处有定义$\}$.

$= \bigcup u$ 为开集.

$\langle u, f \rangle$

$X \setminus \mathrm{Dom}(\varphi)$ 称为不定点

(indeterminacies)

能否稍微改变 X，使新的映射处处有定义？

$$\begin{tikzcd}
X' \arrow[d] \arrow[dr, bend left] & \\
X \arrow[r, dashed] & Y
\end{tikzcd}$$

定义 2. 设 X, Y 为代数簇.

$\varphi: X \dashrightarrow Y$ 有理

称 φ 是双有理的，若 $\exists\ \psi: Y \dashrightarrow X$，

s.t. $\psi \circ \varphi = \mathrm{Id}_X$，$\varphi \circ \psi = \mathrm{Id}_Y$

在有定义点上

基本事实：

X 与 Y 双有理等价

$\Leftrightarrow$ X 与 Y 有同构的非空开集

$\Leftrightarrow$ $\mathbb{C}(X)$，$\mathbb{C}(Y)$ 同构.（作为 $\mathbb{C}$ 代数）

$$\begin{tikzcd} X' \arrow[r, "\Phi'"] \arrow[d, "\pi_X"'] & Y' \arrow[d, "\pi_Y"] \\ X \arrow[r, "\Phi"] & Y \end{tikzcd}$$

爆破：Blow up.

(1) $\pi_x : \mathbb{P}^n \setminus \{x\} \to \mathbb{P}^{n-1}$

$x = [1, \cdots, 0]$

$[X_0, \cdots, X_n] \mapsto [X_1, \cdots, X_n]$

考虑 π 的图像在 $\mathbb{P}^n \times \mathbb{P}^{n-1}$ 闭包

$$X = \{([x_0,\cdots,x_n],[y_1,\cdots,y_n]) \in \mathbb{P}^n \times \mathbb{P}^{n-1},$$

$$[x_0,\cdots,x_n] \neq [1,\cdots,0]$$

$$[x_1,\cdots,x_n] = [y_1,\cdots,y_n]\}$$

$$\supset \{([x_0,\cdots,x_n],[y_1,\cdots,y_n]) \in \mathbb{P}^n \times \mathbb{P}^{n-1},$$

$$x_i y_j - y_j x_i = 0, \ \forall 1 \le i,j \le n\}$$

$P_1 : X \to \mathbb{P}^n$, $P_2 : X \to \mathbb{P}^{n-1}$

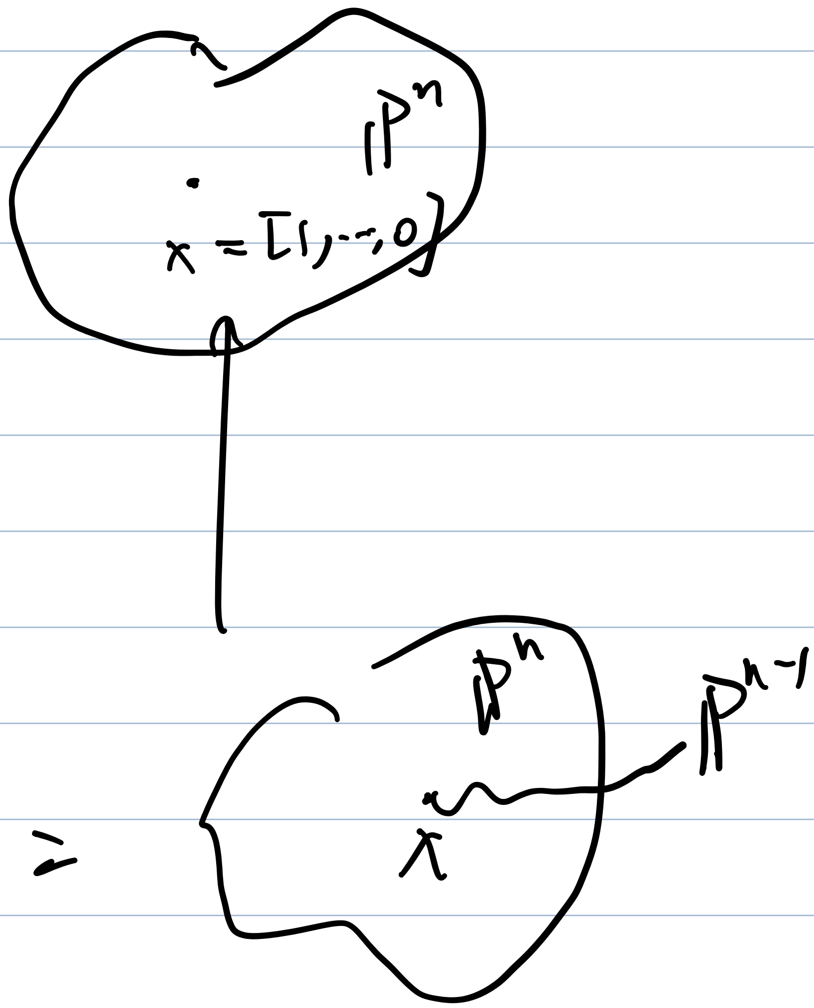
(i) $[x_0,\cdots,x_n] \neq [1,\cdots,0]$ 时

$P_1^{-1}([x_0,\cdots,x_n])$ 为单点集

$n-1$

(ii) $p_1^{-1}([1,\cdots,0]) \cong \mathbb{P}$

$X =$



我们称 X 为 $\mathbb{P}^n$ 关于 x 的爆破

$$X = Bl_x \mathbb{P}^n$$

(2) $Bl_0 \mathbb{C}^n = \{((X_1,\cdots,X_n),[Y_1,\cdots,Y_n])$

$\in \mathbb{C}^n \times \mathbb{P}^{n-1}$

$x_i y_j = x_j y_i \}$

(i) $x \neq 0$

$p_1^{-1}(x) = \{(x, \mathbb{C}x)\}$

(ii) $x = 0$

$p_1^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}$

爆破为局部操作

可对复流形定义.

爆破为一新的复流形

设 $X \subseteq \mathbb{P}^n$ 为代数簇, $x \in X$

$$\begin{tikzcd} X \arrow[r, hook] & \mathbb{P}^n \\ & \mathrm{Bl}_x \mathbb{P}^n \arrow[u, "p_1"'] \end{tikzcd}$$

$$\mathrm{Bl}_x X = \overline{p_1^{-1}(X - \{x\})}$$